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Basics of Laser Interferometer Sensing and Control

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Topics

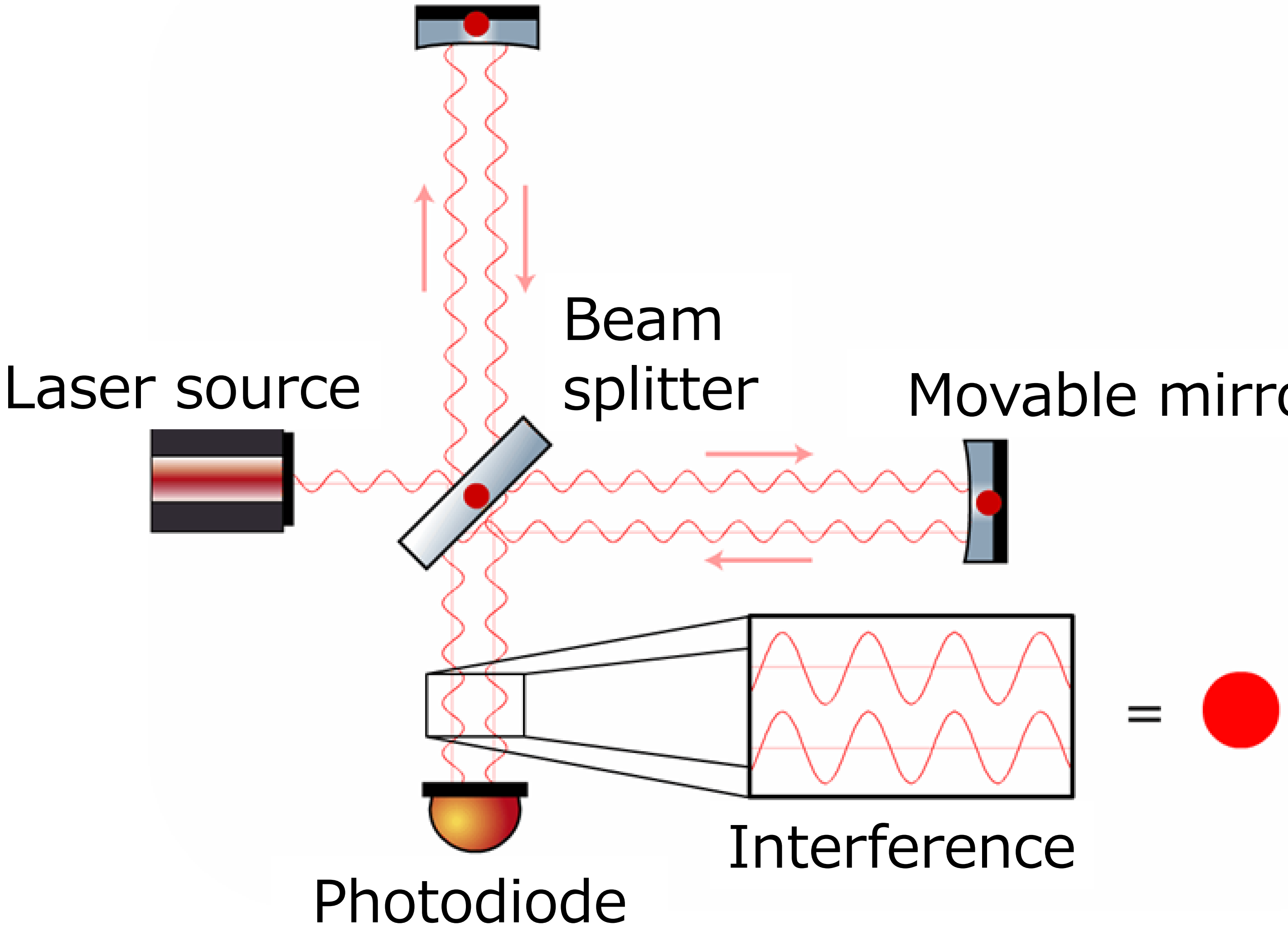
- **Michelson interferometer**
- **Linear system**
- **Power spectrum**
- **Transfer function**
- **Feedback system**
- **Quantum noise**

Further Reading

- Eric D. Black, Ryan N. Gutenkunst, *An introduction to signal extraction in interferometric gravitational wave detectors*, Am. J. Phys. 71, 365 (2003)
- Peter R. Saulson, *Fundamentals of Interferometric Gravitational Wave Detectors* (World Scientific, 1994)
- 中村卓, 三尾典克, 大橋正健 編著『重力波をとらえる』(京都大学学術出版会, 1998)
- 日野幹雄『スペクトル解析』(朝倉書店, 2010)
- 片山徹『フィードバック制御の基礎』(朝倉書店, 2002)
- 東京大学 物理学実験II ブラウン運動 テキスト

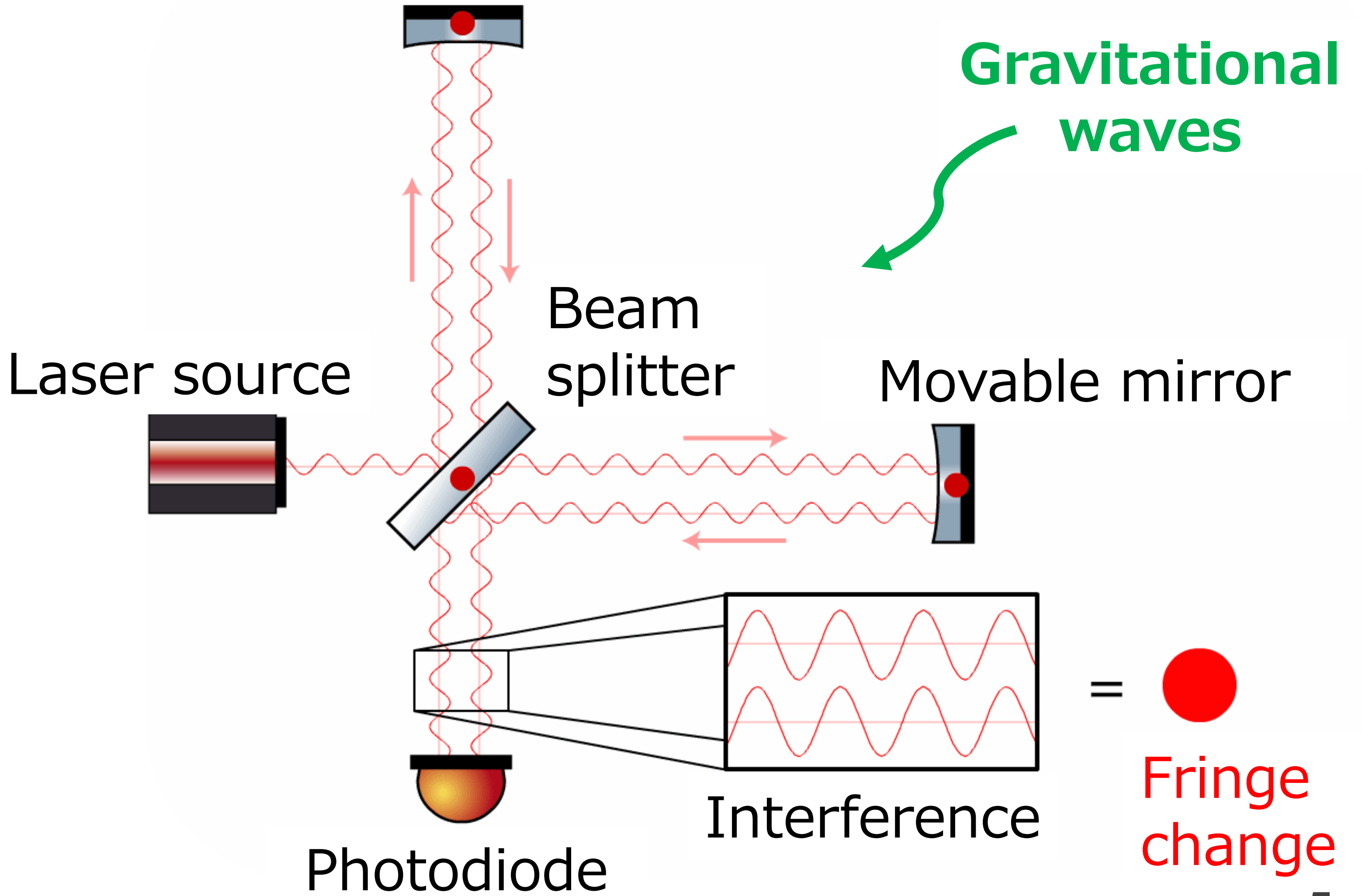
Michelson Interferometer

- Measures differential arm length change



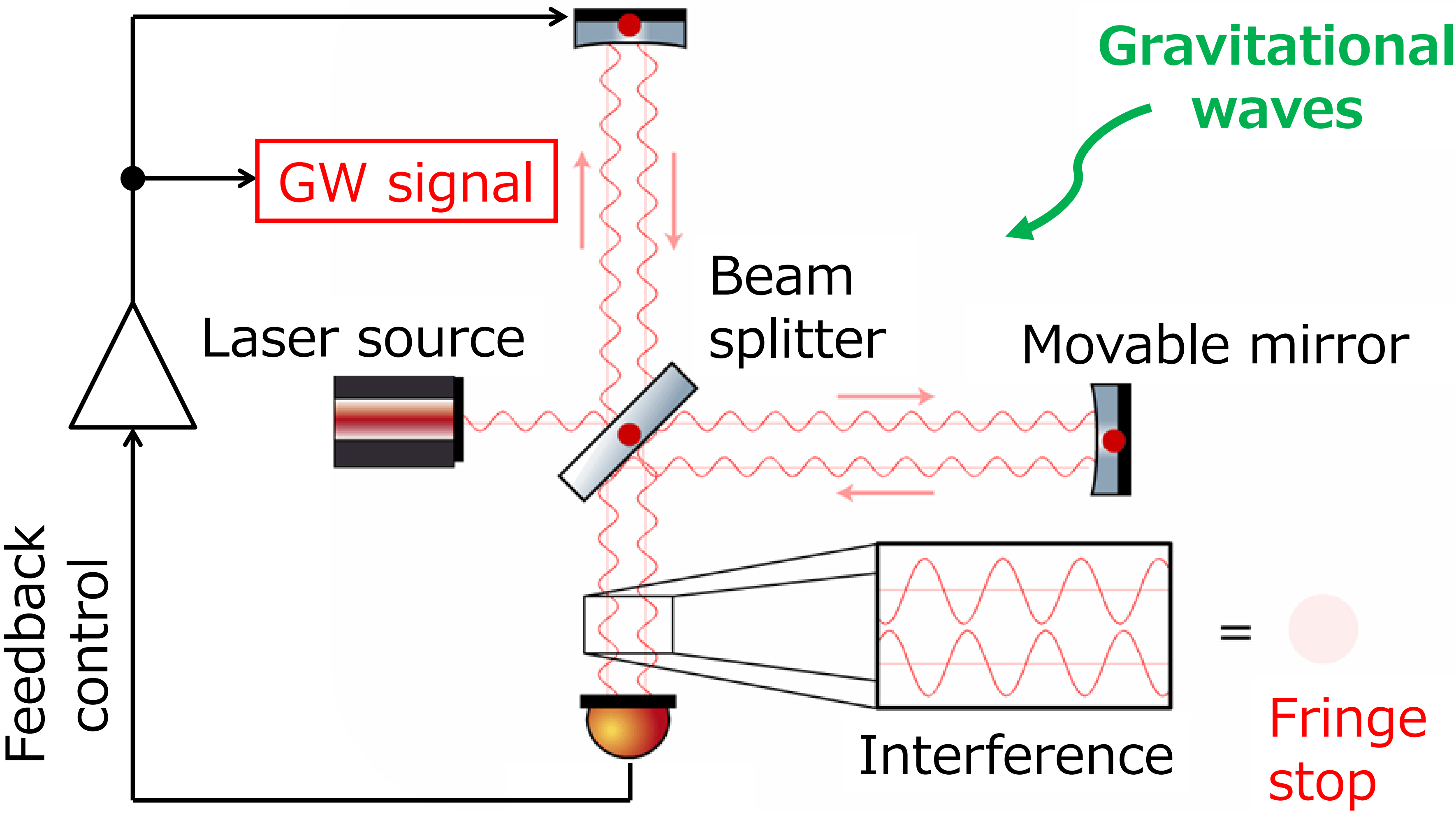
Michelson Interferometer

- Measures differential arm length change



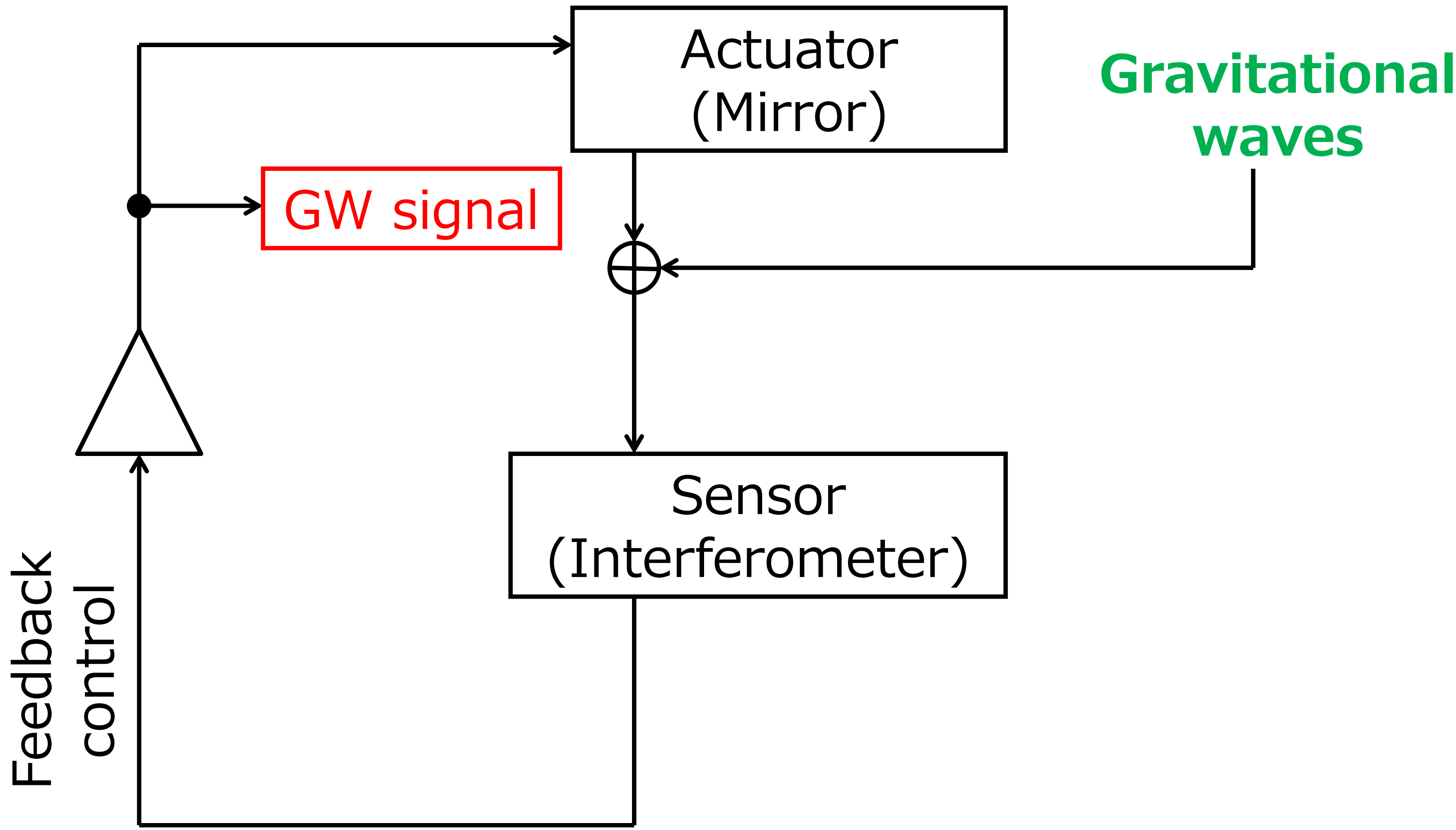
Interferometer Control

- Control mirror positions to “lock” the fringe



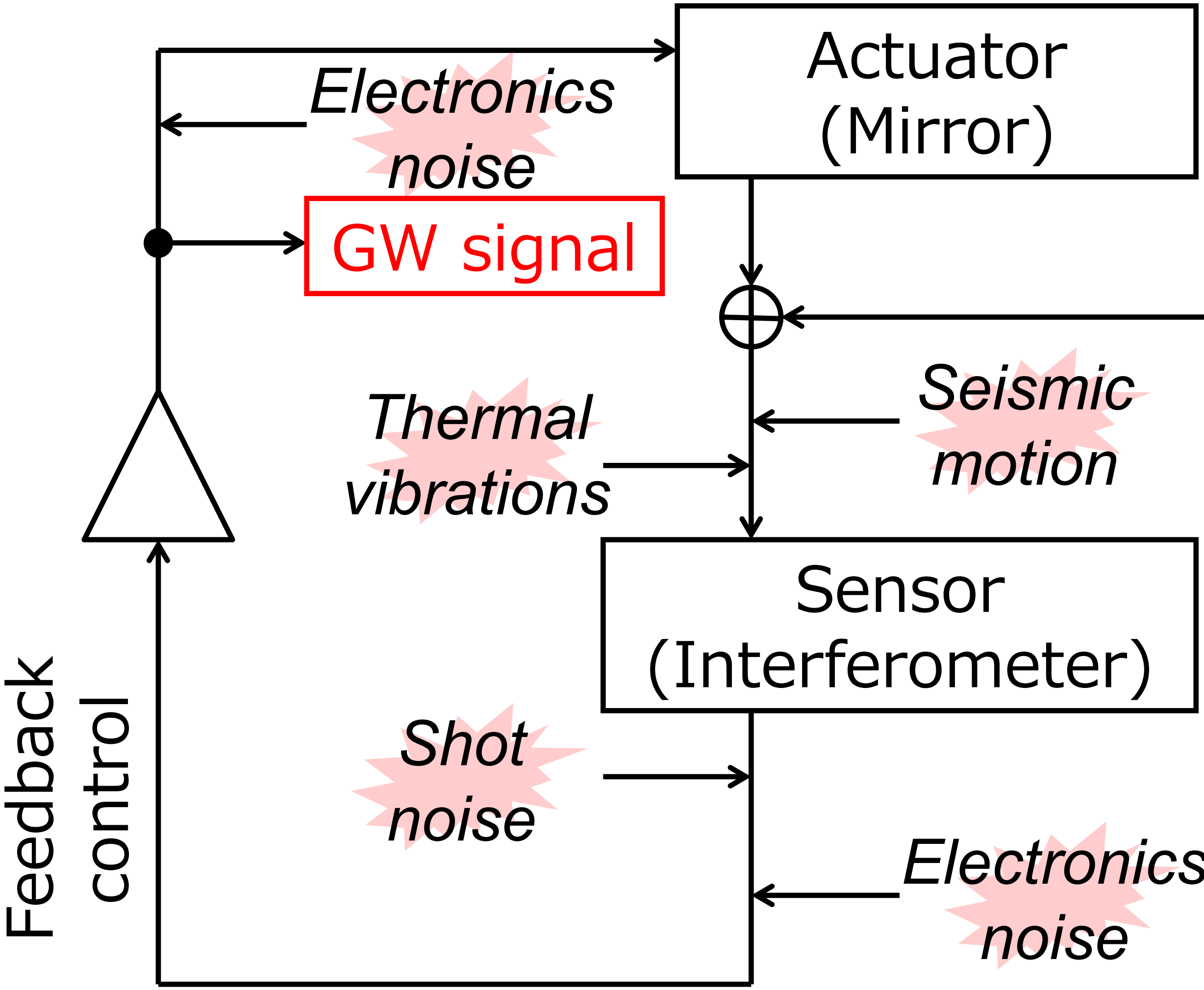
Linear System

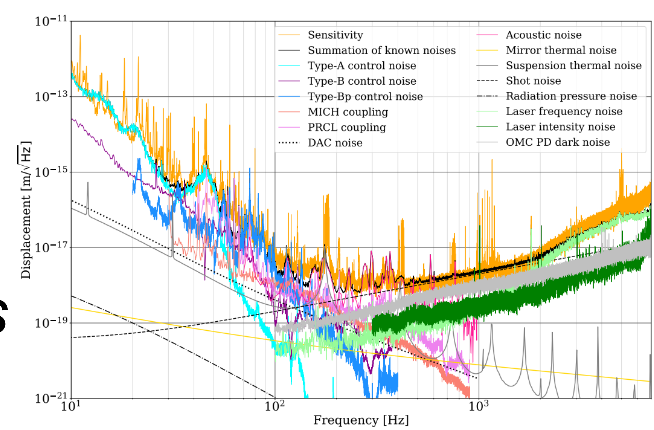
- Control makes the system linear



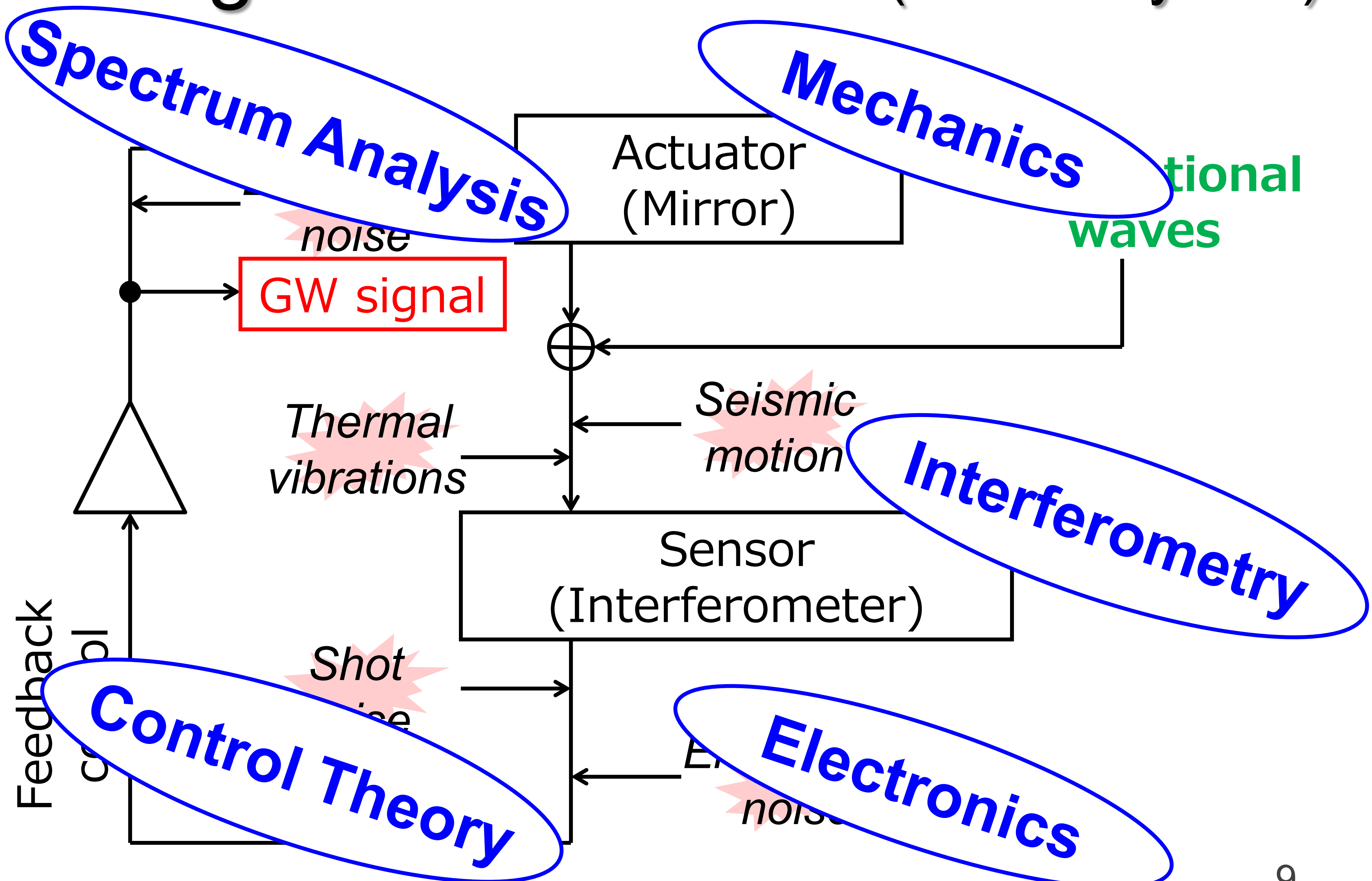
Noises and Sensitivity

- Noises in the loop limits the sensitivity



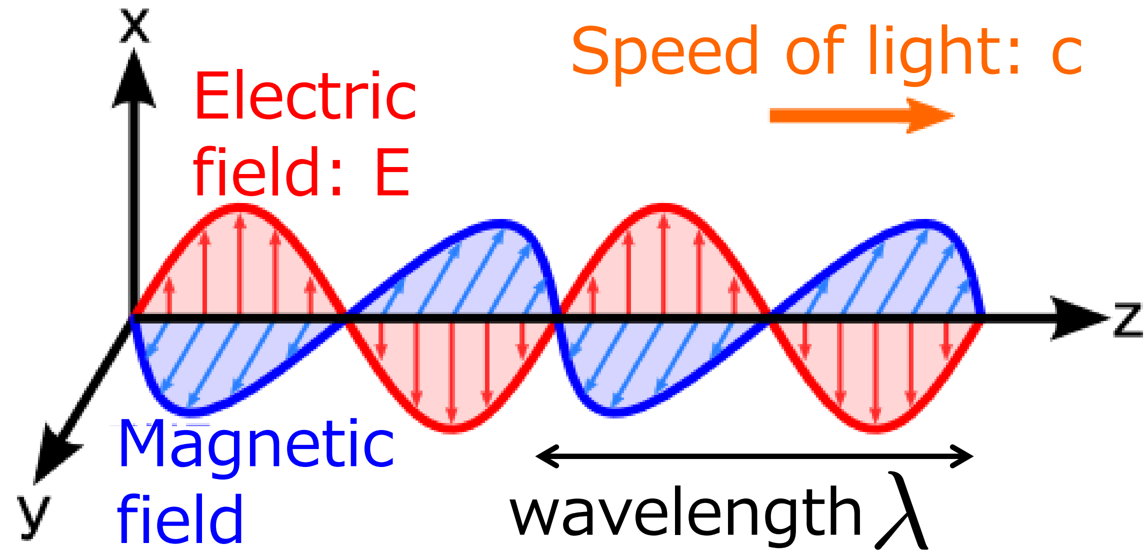


Things You Will Learn (over the years)



Laser Beam

- Electro-magnetic waves



- Electric field can be written as

$$E = E_0 e^{i(\omega t - \phi)}$$

amplitude

angular
frequency
of laser

phase

$$\omega = \frac{2\pi c}{\lambda}$$

$$\phi = \frac{2\pi L}{\lambda}$$

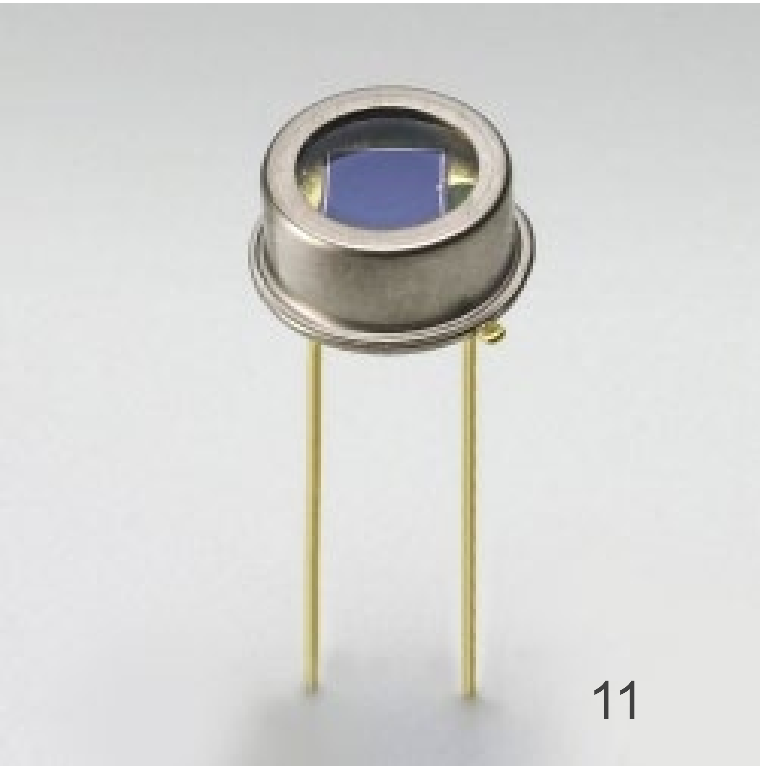
phase at
distance L

Photodiodes

- Photodiodes (PDs)
  Convert photons into electrons
  Detects light power (square of amplitude)

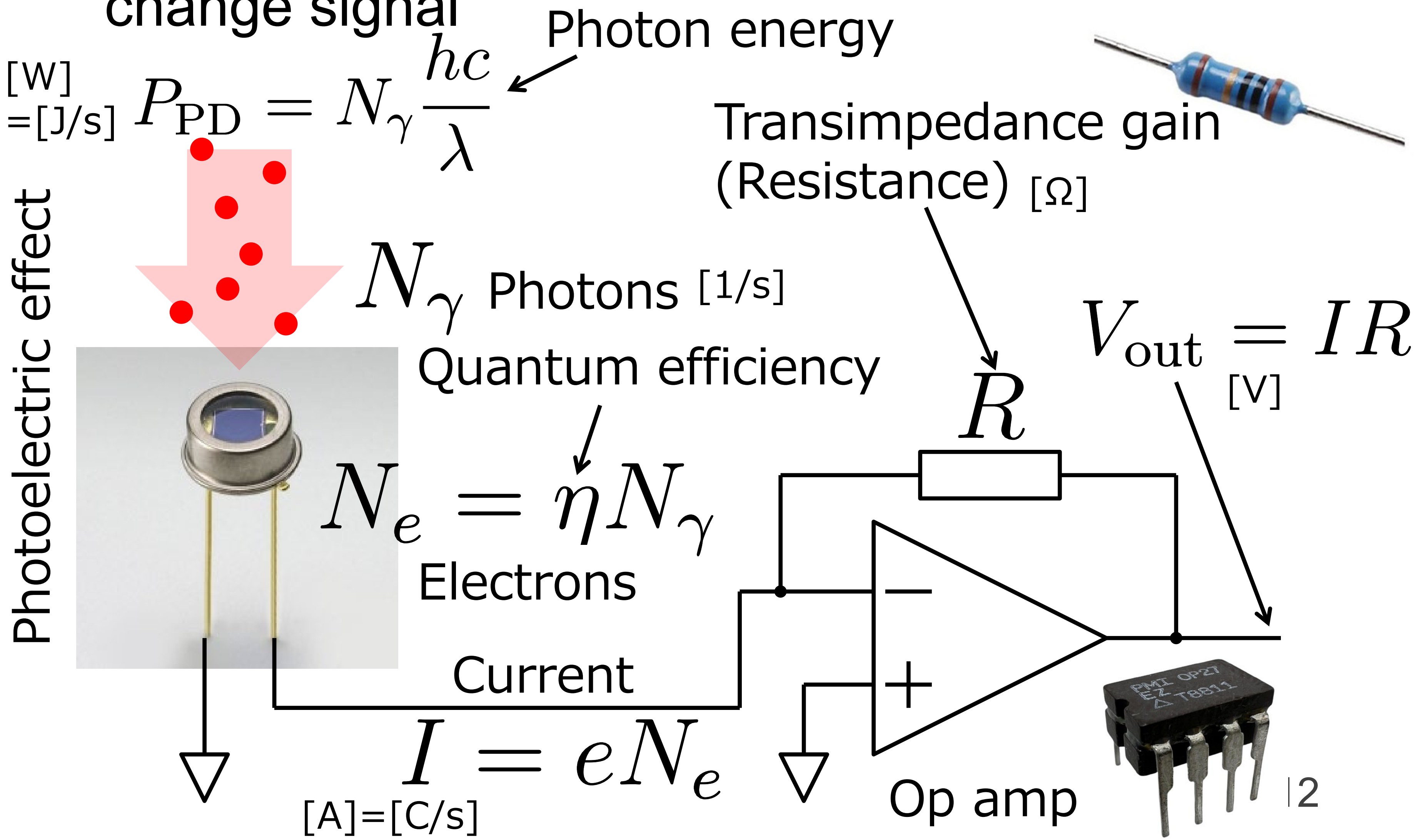
$$P \propto |E|^2 = E_0^2$$

We can only detect power change
Phase change cannot be detected directly



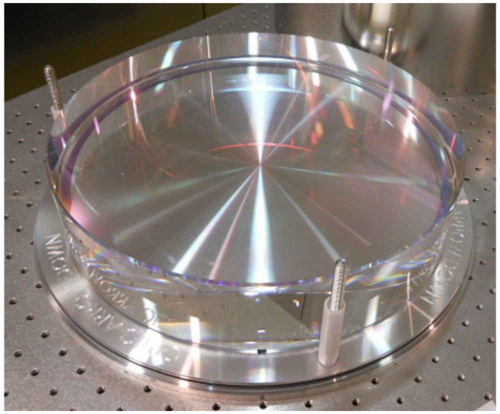
Transimpedance Amplifier

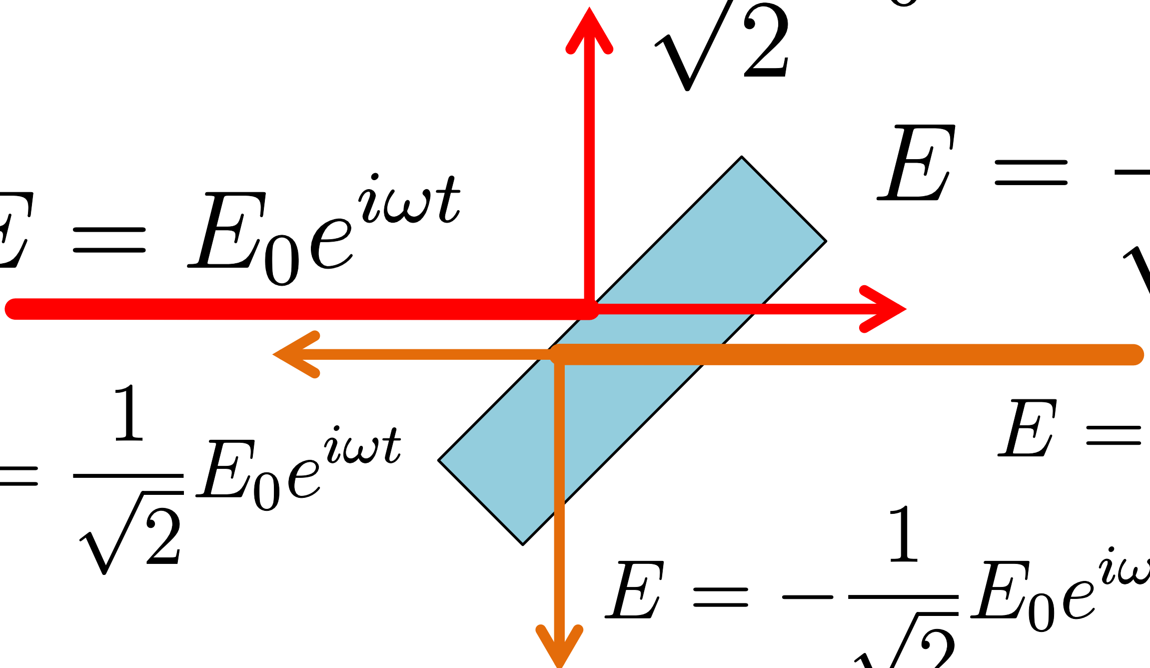
- Power change signal will be converted into voltage change signal



Beam Splitter

- Split beam in two
- Half in power, $1/\sqrt{2}$ in amplitude
- Sign flip in back reflection





$$E = \frac{1}{\sqrt{2}} E_0 e^{i\omega t}$$

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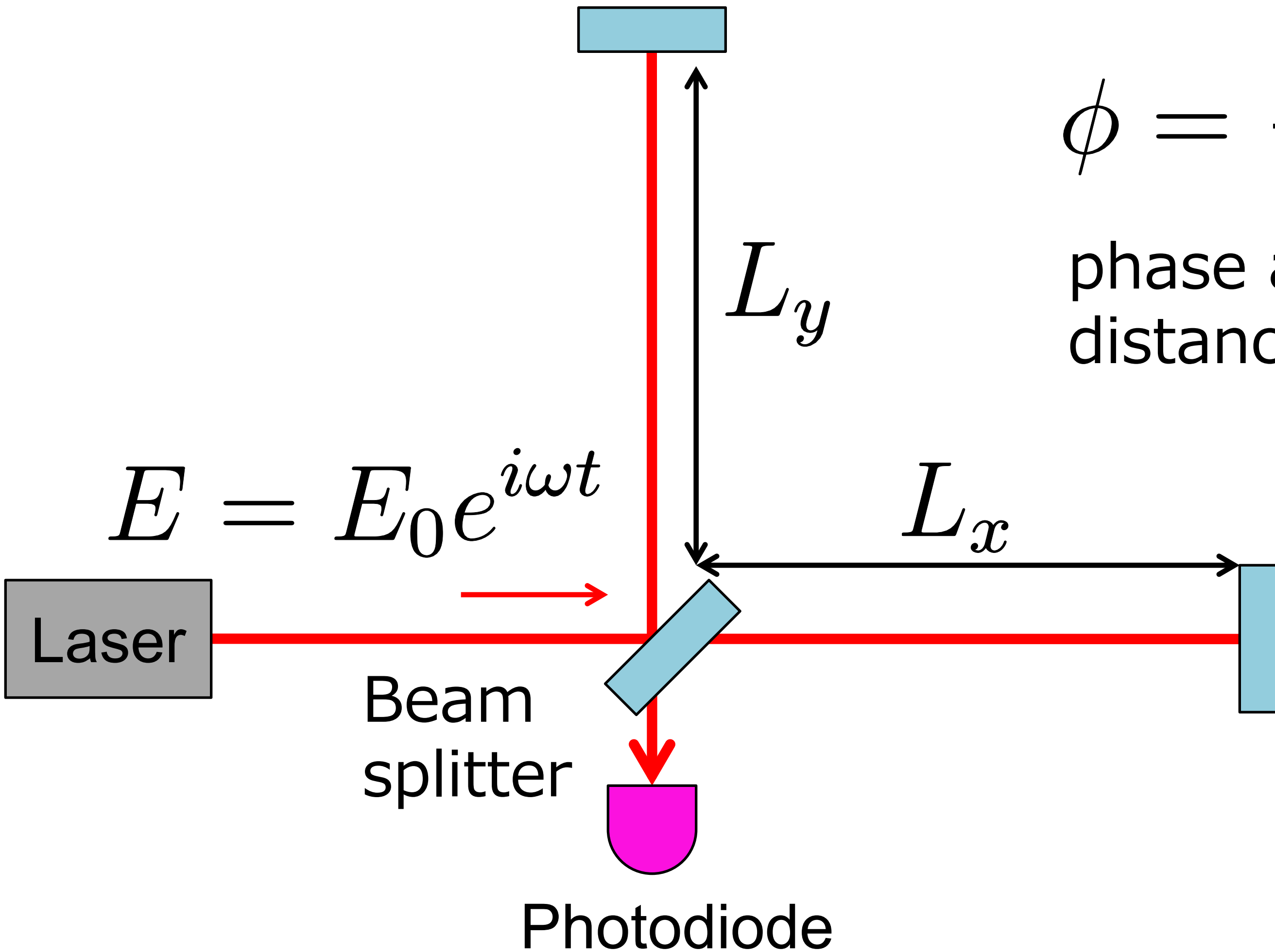
$$E = \frac{1}{\sqrt{2}} E_0 e^{i\omega t}$$

$$E = E_0 e^{i\omega t}$$

$$E = -\frac{1}{\sqrt{2}} E_0 e^{i\omega t}$$

Output of Michelson Interferometer

- What is the power detected at the photodiode?



$$\phi = \frac{2\pi L}{\lambda}$$

phase at distance L

Output of Michelson Interferometer

- What is the power detected at the photodiode?

From Y-am From X-arm

$$P_{\mathrm{PD}} = \left| \frac{1}{2} E_0 e^{i\left(\omega t - \frac{4\pi L_y}{\lambda}\right)} - \frac{1}{2} E_0 e^{i\left(\omega t - \frac{4\pi L_x}{\lambda}\right)} \right|^2$$

$$= \frac{1}{4} |E_0|^2 \left| e^{-i\frac{4\pi L_y}{\lambda}} - e^{-i\frac{4\pi L_x}{\lambda}} \right|^2$$

$$= \frac{1}{2} P_0 \left(1 - \cos \frac{4\pi L_-}{\lambda} \right)$$

Input power

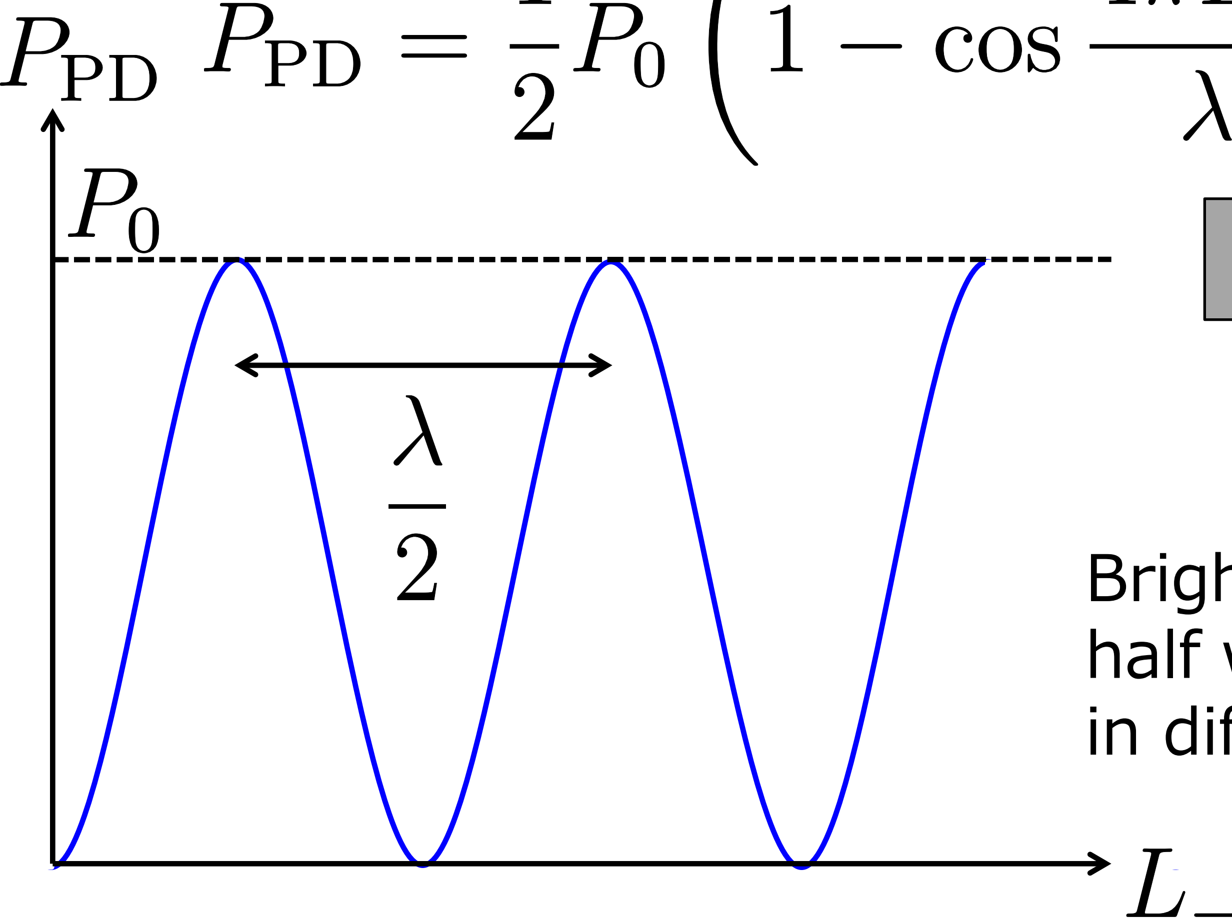
$$L_- = L_y - L_x$$

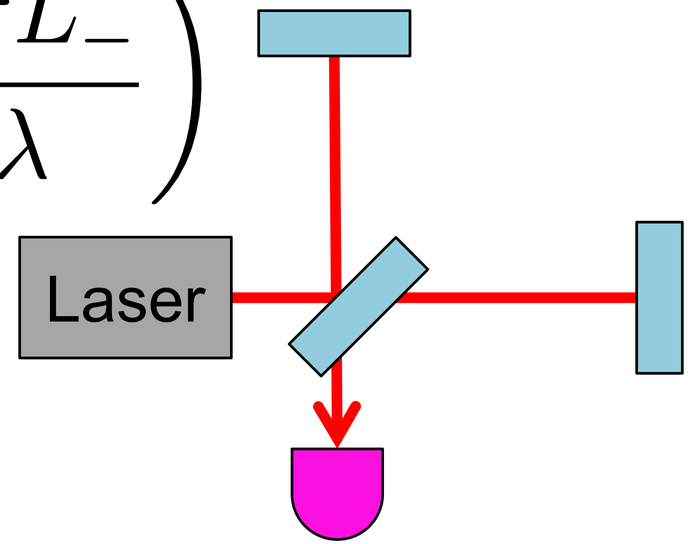
Differential arm length

Output of Michelson Interferometer

- Power changes with differential arm length change (interference)

$$P_{\mathrm{PD}} = \frac{1}{2} P_0 \left(1 - \cos \frac{4\pi L_-}{\lambda} \right)$$



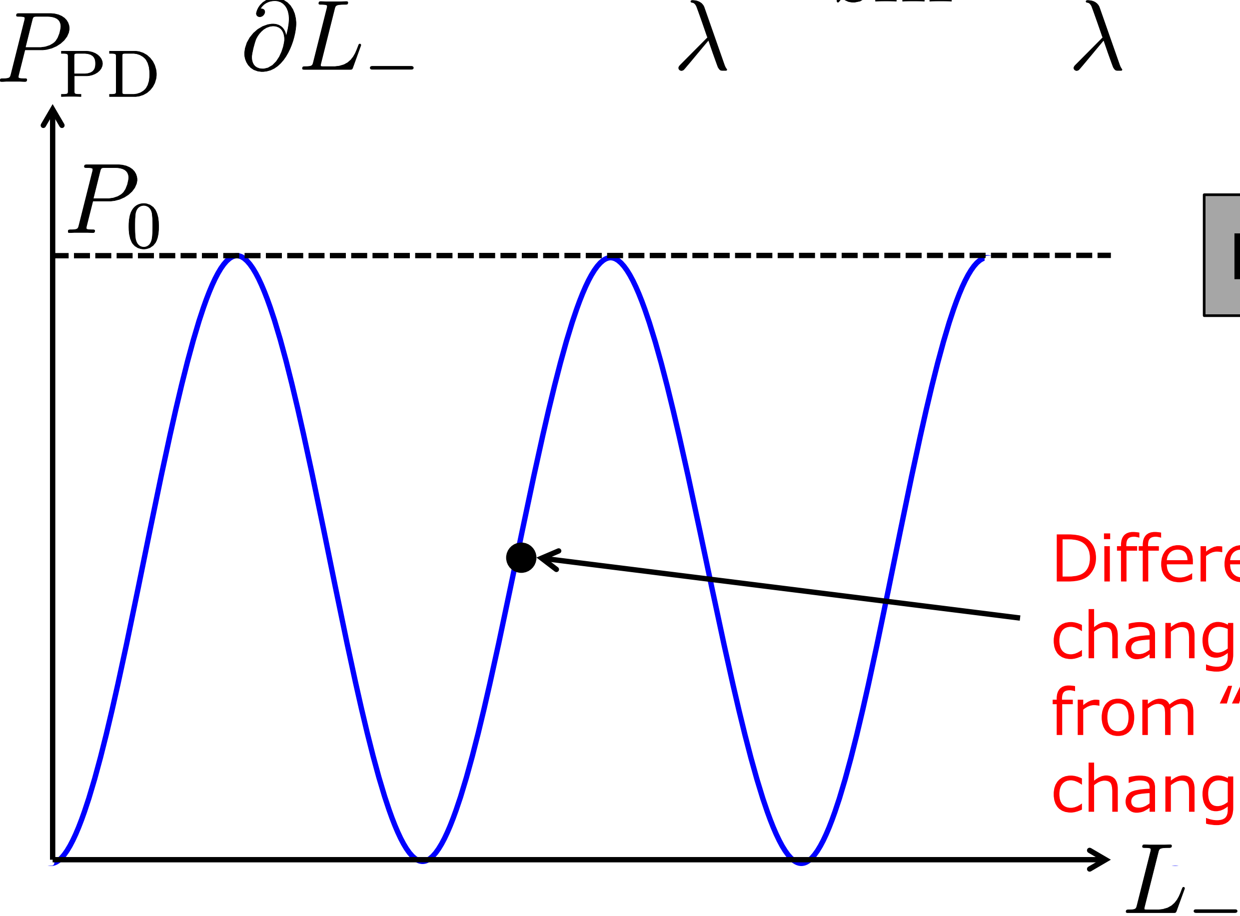


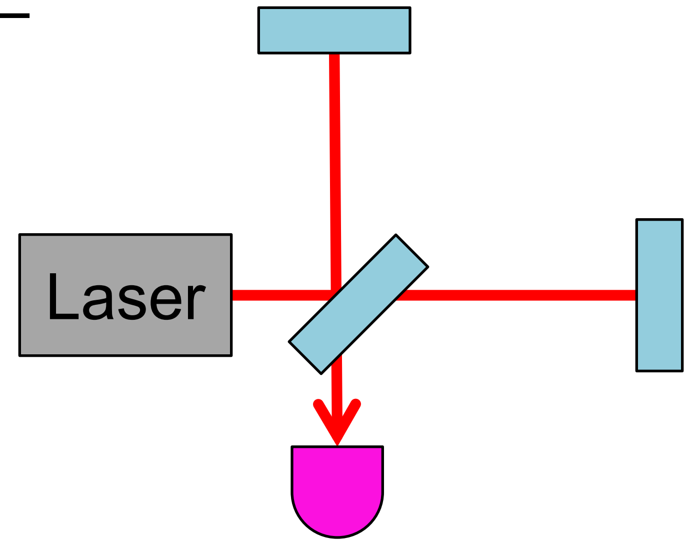
Bright fringe in every half wavelength change in differential arm length

Output of Michelson Interferometer

- Ratio between power change and length change

$$\frac{\partial P_{\mathrm{PD}}}{\partial L_-} = \frac{2\pi P_0}{\lambda} \sin \frac{4\pi L_-}{\lambda}$$

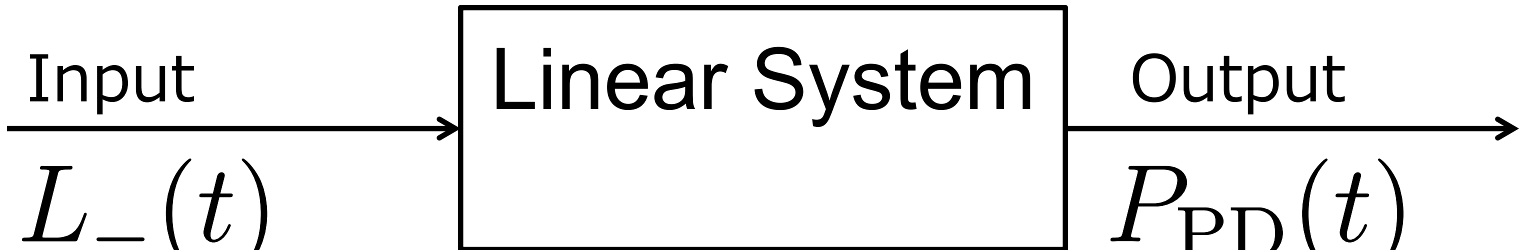




Differential arm length change can be detected from “linear” power change at the photodiode

Linear System

- System that has linear input and output relations



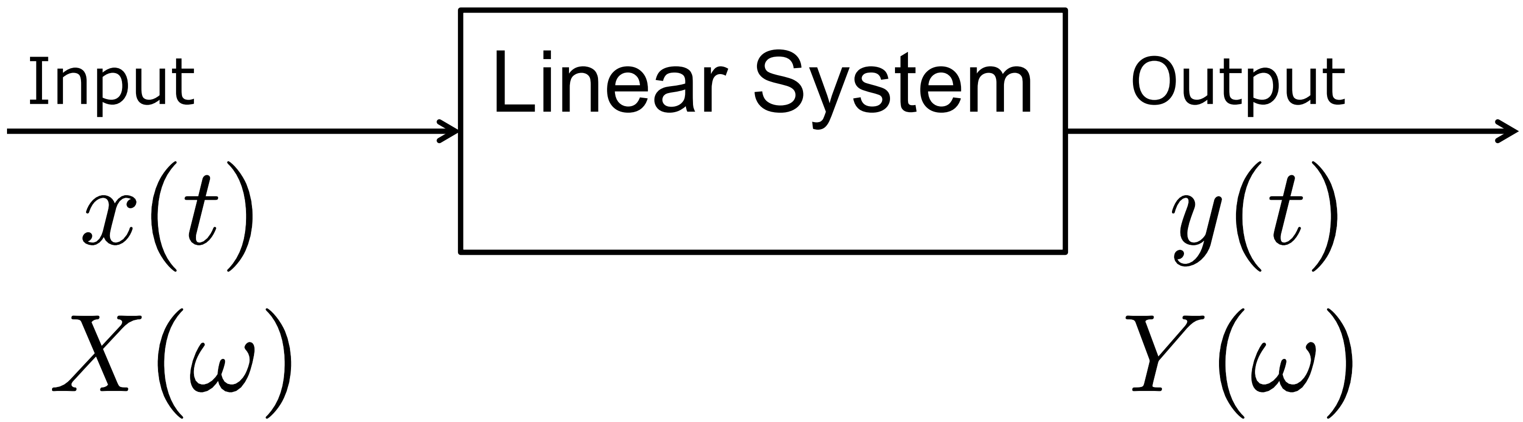
$L_-(t)$ $\quad$ $P_{\mathrm{PD}}(t)$

$aL_-(t)$ Homogeneity $aP_{\mathrm{PD}}(t)$

$L_- + L_{\mathrm{seis}}(t)$ Additivity $P_{\mathrm{PD}}(t) + P_{\mathrm{seis}}(t)$

Fourier Transform

- Handy to think in frequency domain



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$$

Power Spectrum Density

- Handy to think in frequency domain

Input → **Linear System** → Output

$x(t)$ [m] $\qquad\qquad$ $y(t)$ [W]

$X(\omega)$ $\qquad\qquad$ $Y(\omega)$

$S_x(\omega)$ [m²/Hz] $\quad$ Power Spectrum Density $\quad$ $S_y(\omega)$ [W²/Hz]

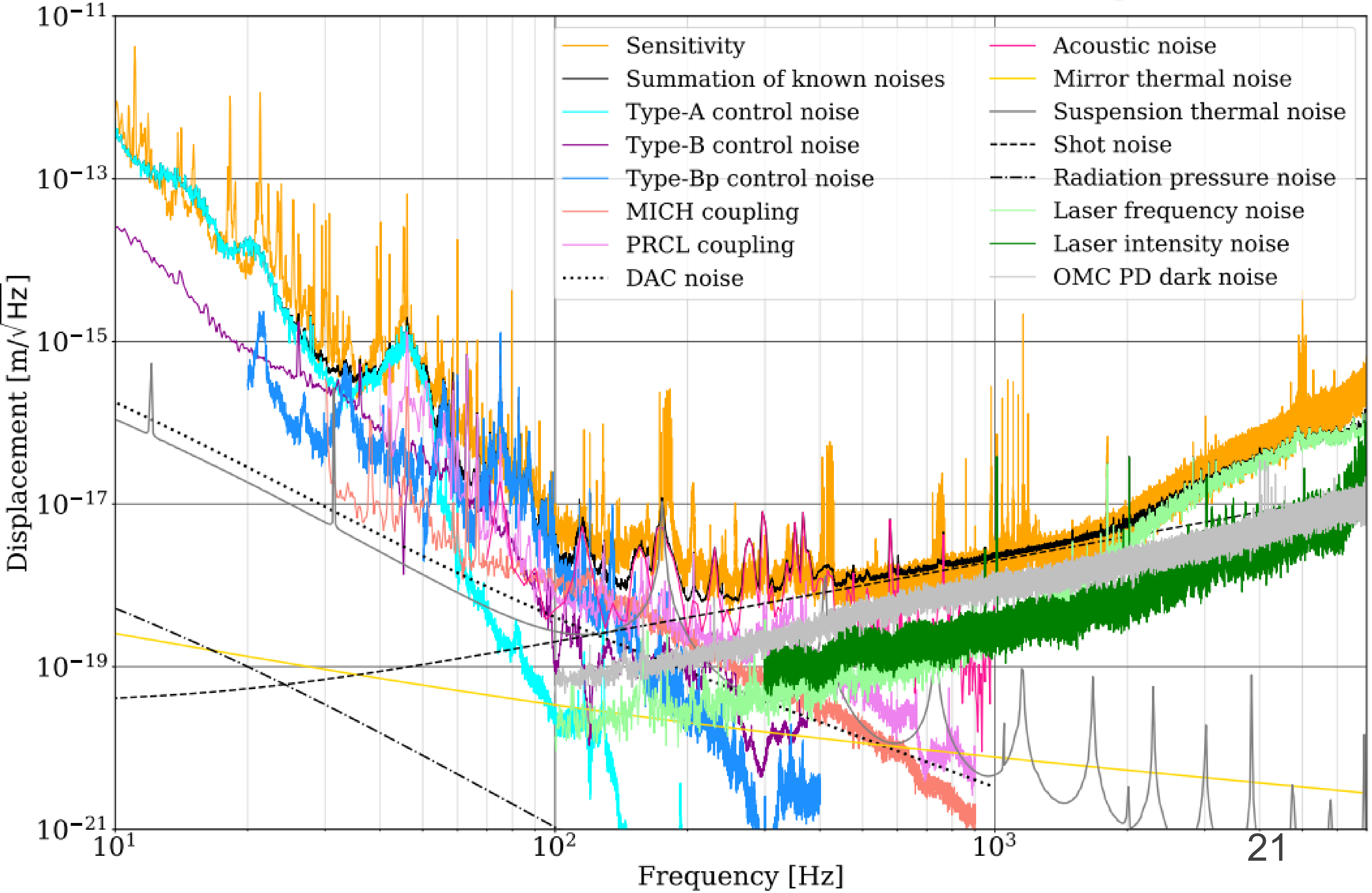
$\sqrt{S_x}(\omega)$ [m/√Hz] $\quad$ Amplitude Spectrum Density $\quad$ $\sqrt{S_y}(\omega)$ [W/√Hz]

$$\boxed{S(\omega) = \lim_{T\to\infty} \left\langle \frac{|X_T(\omega)|^2}{T} \right\rangle}$$

Fourier transform of

$$x_T(t) = \begin{cases} x(t) & |t| \le T/2 \\ 0 & |t| < T/2 \end{cases}$$

Power Spectrum Density



Transfer Function

- Ratio of input and output

Input $\longrightarrow$ **Linear System** $H(\omega)$ $\longrightarrow$ Output

$x(t)$ [m] | $y(t)$ [W]

$X(\omega)$ | $Y(\omega)$

$S_x(\omega)$ [m²/Hz] | Power Spectrum Density | $S_y(\omega)$ [W²/Hz]

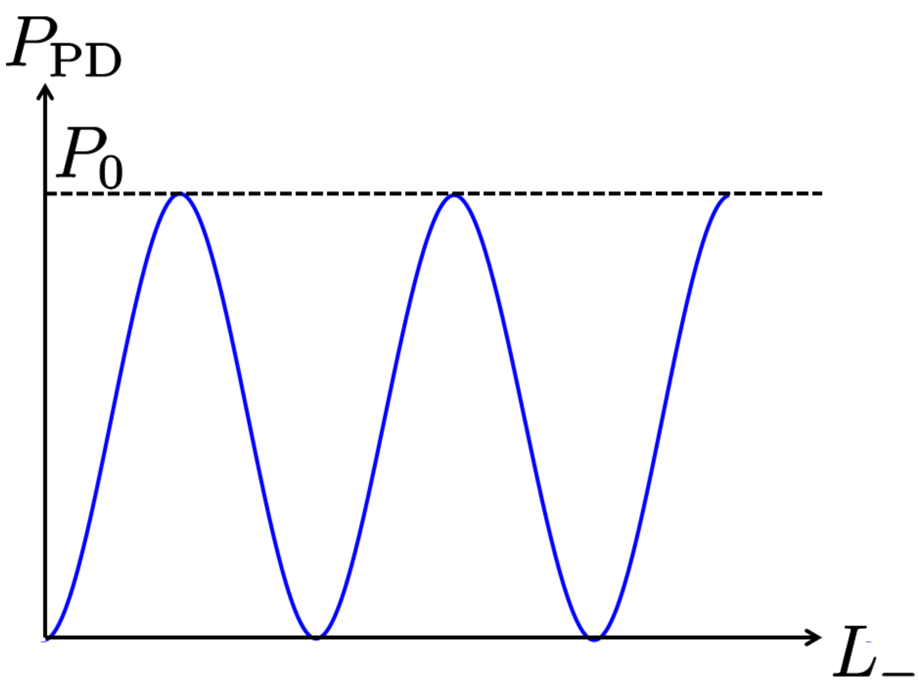
$\sqrt{S_x}(\omega)$ [m/√Hz] | Amplitude Spectrum Density | $\sqrt{S_y}(\omega)$ [W/√Hz]

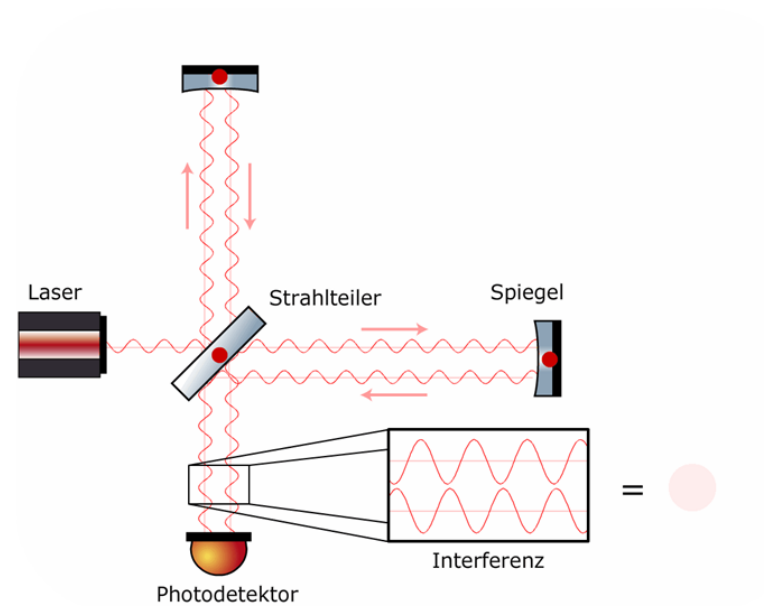
$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}} \Rightarrow \sqrt{S_y}(\omega) = |H(\omega)|\sqrt{S_x}(\omega)$$

Transfer Function of Interferometer

- Constant if fringe is fixed

$$H_{\mathrm{IFO}}(\omega) = \frac{P_{\mathrm{PD}}(\omega)}{L_{-}(\omega)} = \frac{2\pi P_0}{\lambda} \sin \frac{4\pi L_{-}}{\lambda}$$





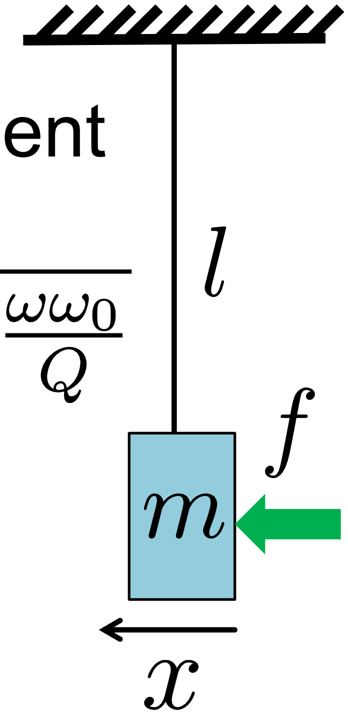
Transfer Function of Actuator

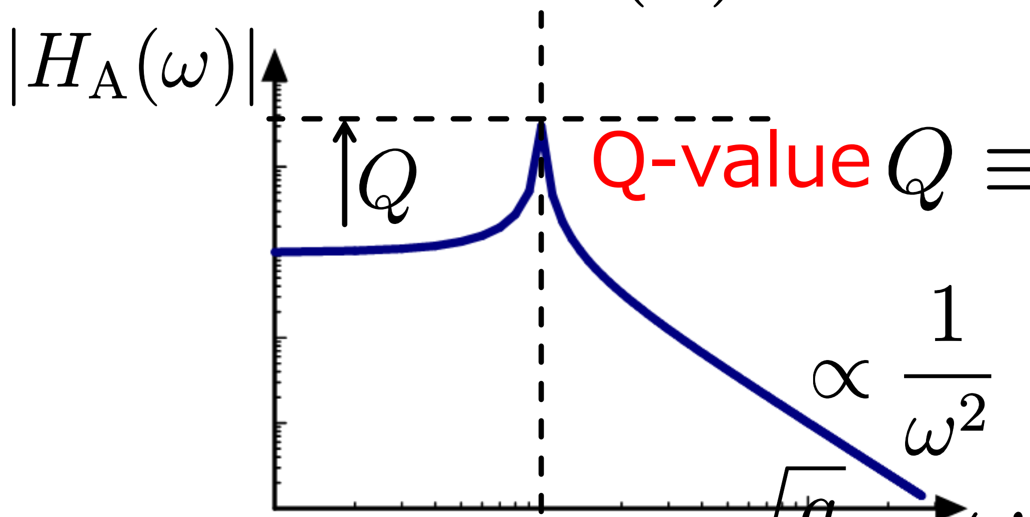
- Equation of motion of a suspended mirror

$$m\ddot{x} = -\gamma\dot{x} - \frac{mg}{l}x + f$$

- Transfer function from force to displacement

$$H_{\mathrm{A}}(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{m}\frac{1}{-\omega^2 + \omega_0^2 + i\frac{\omega\omega_0}{Q}}$$



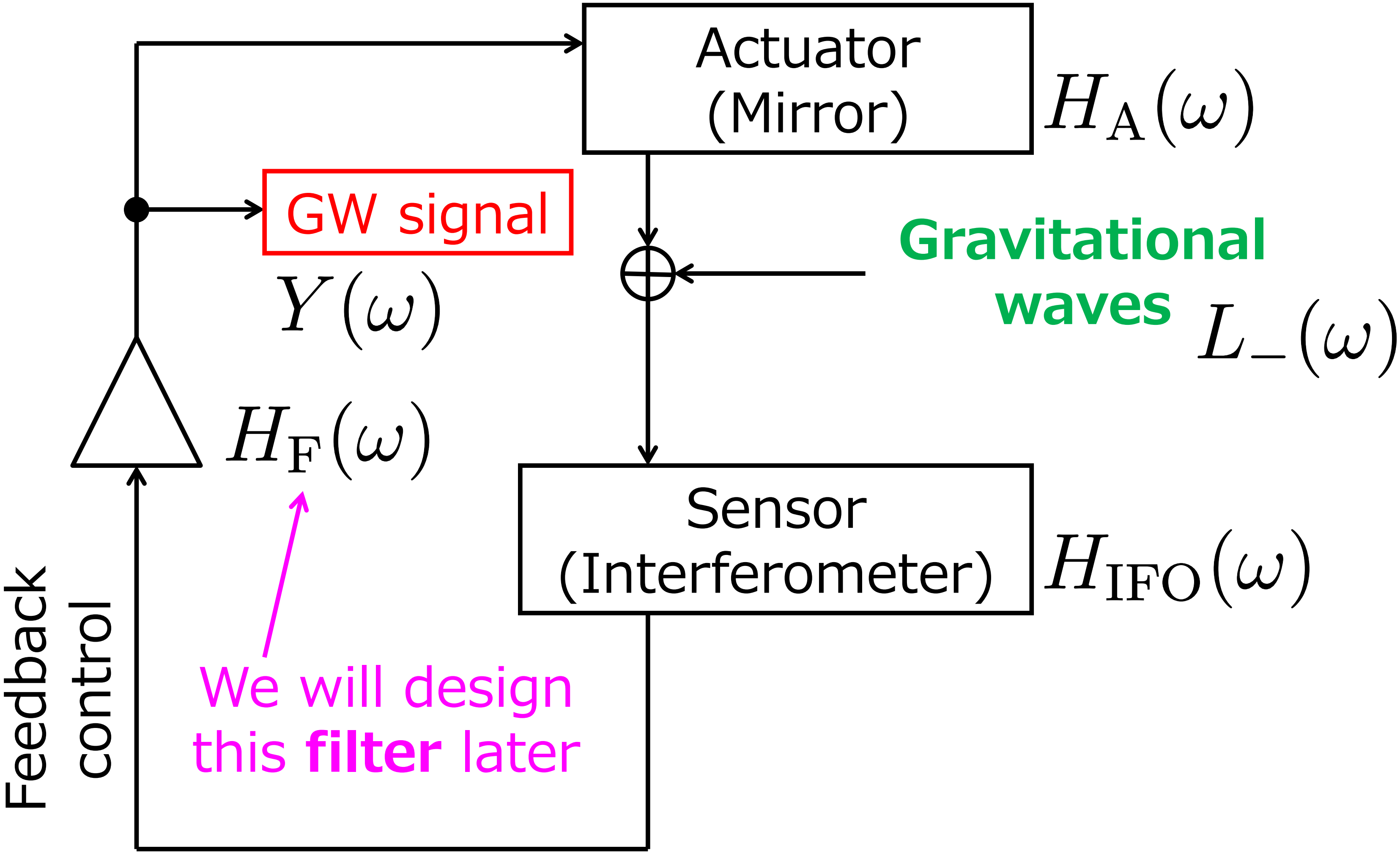


Q-value $Q \equiv \frac{\omega_0 m}{\gamma}$

Resonant frequency $\omega_0 \equiv \sqrt{\frac{g}{l}}$

Feedback Control Loop

- Now we can analyze the loop in frequency domain



Feedback Control Loop

- Open loop transfer function G(ω) is important

$$Y(\omega) = H_{\mathrm{F}}(\omega)H_{\mathrm{IFO}}(\omega)\left(L_{-}(\omega) - H_{\mathrm{A}}(\omega)Y(\omega)\right)$$

$$= \frac{H_{\mathrm{F}}(\omega)H_{\mathrm{IFO}}(\omega)L_{-}(\omega)}{1 + G(\omega)}$$

$$G(\omega) \equiv H_{\mathrm{F}}H_{\mathrm{IFO}}H_{\mathrm{A}}$$

- GW signal can be estimated from Y(ω) (feedback signal) if G(ω) >>1

$$L_{-}(\omega) = \frac{1 + G(\omega)}{G(\omega)}H_{\mathrm{A}}(\omega)Y(\omega)$$

$$\simeq H_{\mathrm{A}}(\omega)Y(\omega)$$

Feedback Control Loop

- Open loop transfer function G(ω) is important

$$Y(\omega) = H_{\mathrm{F}}(\omega)H_{\mathrm{IFO}}(\omega)\left(L_{-}(\omega) - H_{\mathrm{A}}(\omega)Y(\omega)\right)$$
$$= \frac{H_{\mathrm{F}}(\omega)H_{\mathrm{IFO}}(\omega)L_{-}(\omega)}{1+G(\omega)}$$

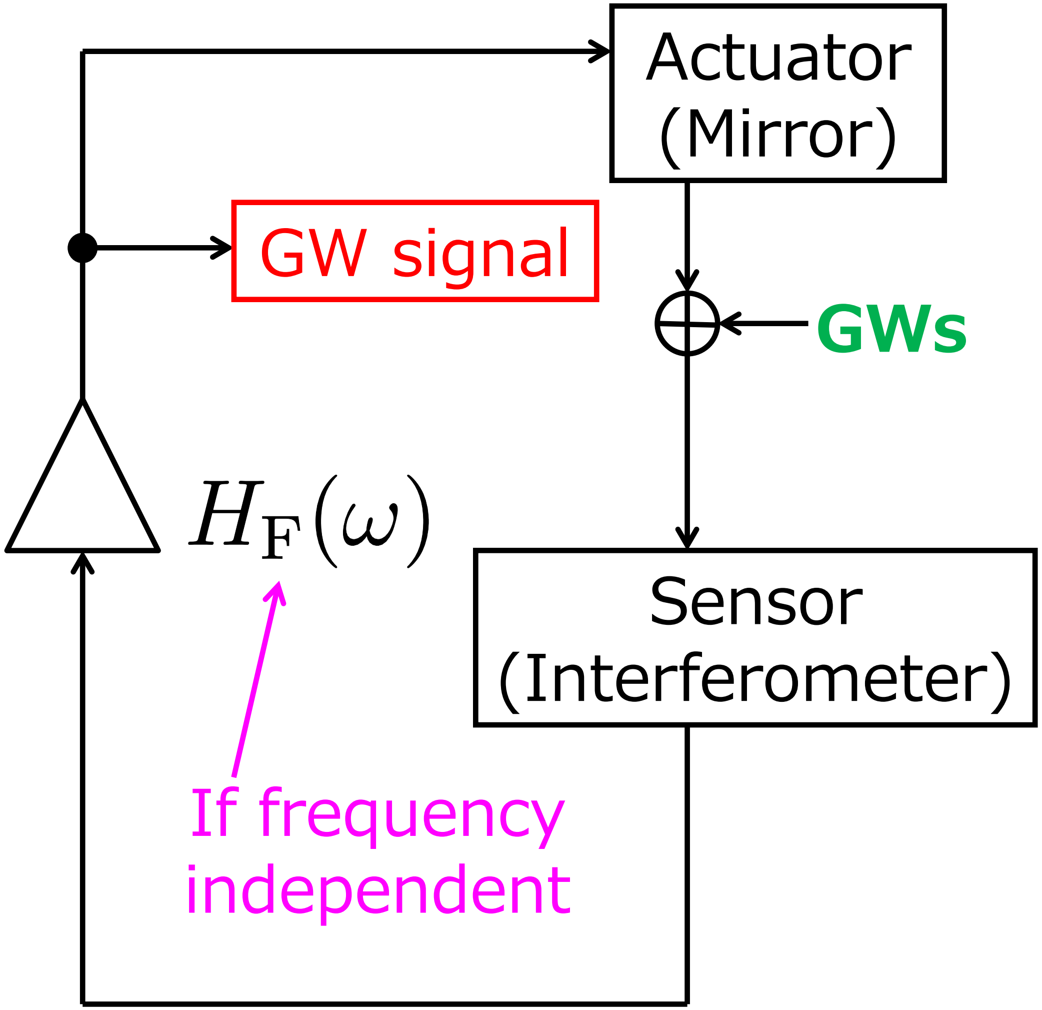
$$G(\omega) \equiv H_{\mathrm{F}}H_{\mathrm{IFO}}H_{\mathrm{A}}$$

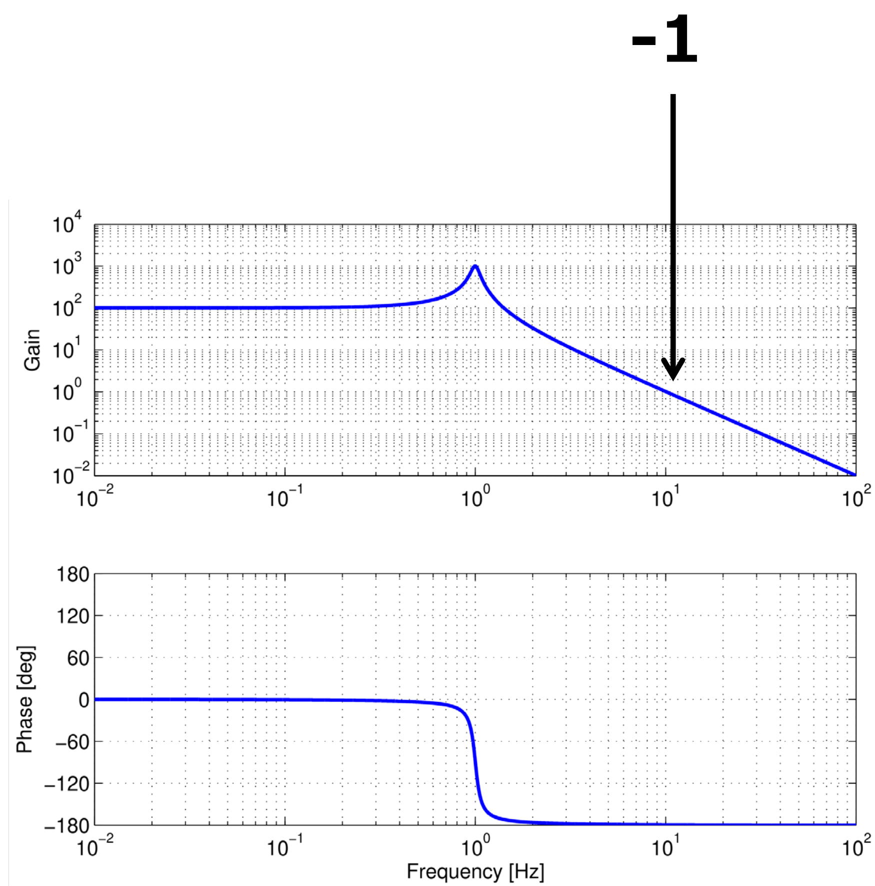
Loop can "oscillate" if G(ω)=-1
→ **Nyquist stability criterion**

Control filter is designed to meet this criterion

Unstable Loop

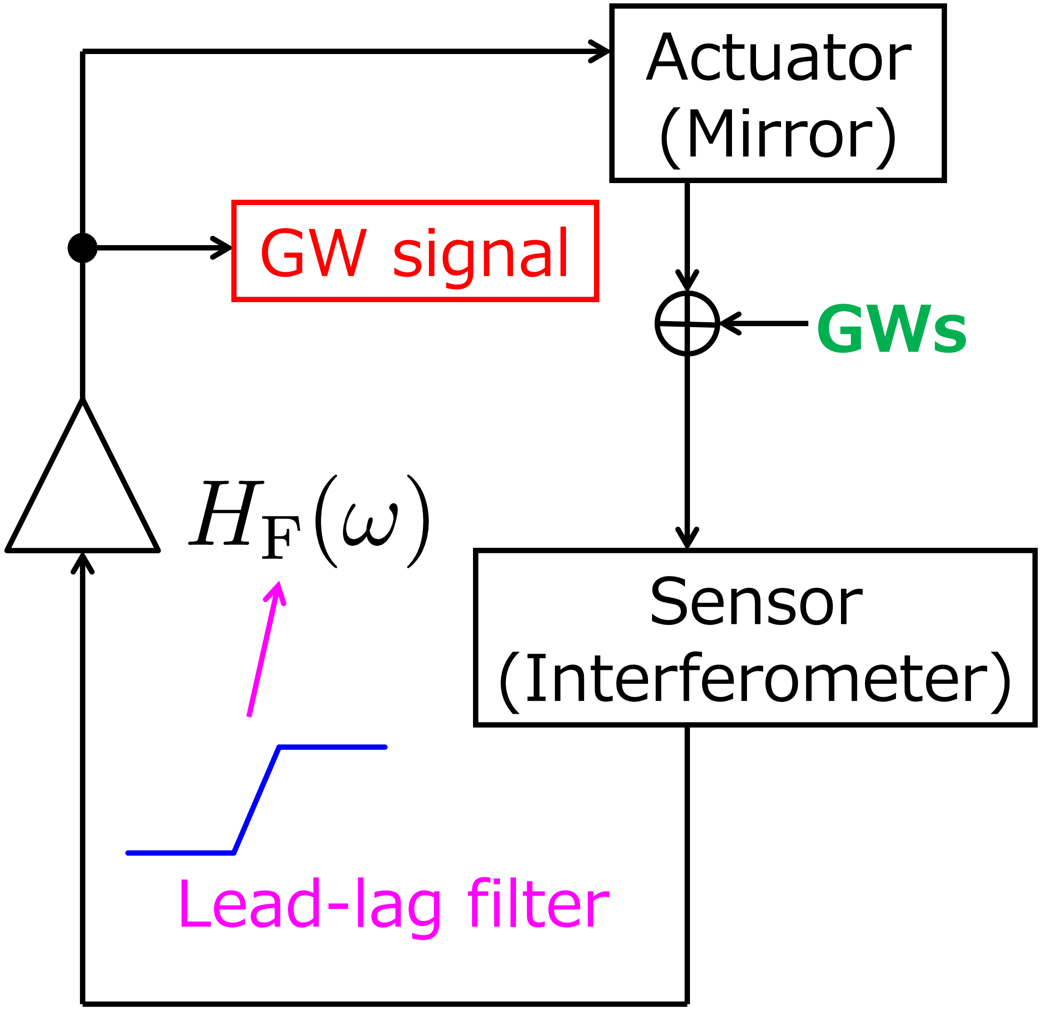
- G(ω)=-1 at unity gain frequency

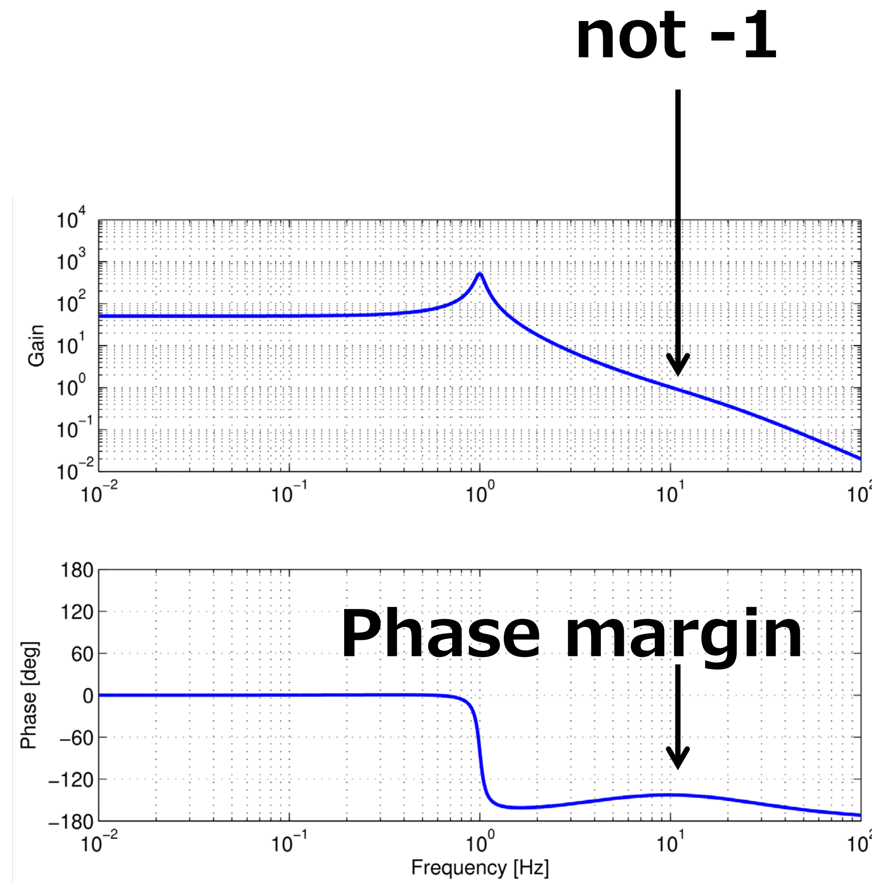




Stable Loop

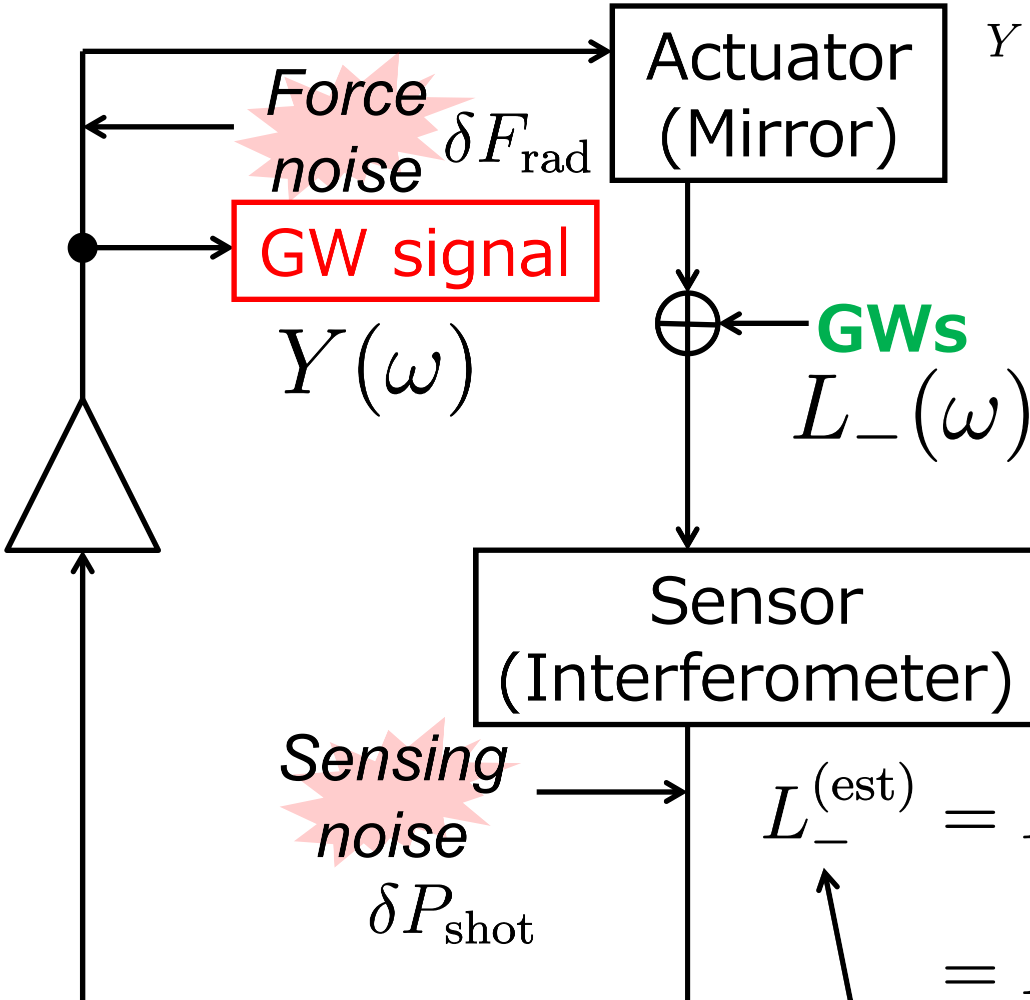
- Phase margin at unity gain frequency





Force Noise and Sensing Noise

- Loop calculation gives noise contributions



$$Y = H_{\mathrm{F}}\left(H_{\mathrm{IFO}}\left(L_- - H_{\mathrm{A}}(Y + \delta F_{\mathrm{rad}})\right) + \delta P_{\mathrm{shot}}\right)$$
$$= \frac{GL_-}{H_{\mathrm{A}}(1+G)} + \frac{G}{1+G}\delta F_{\mathrm{rad}} + \frac{G}{H_{\mathrm{IFO}}H_{\mathrm{A}}(1+G)}\delta F_{\mathrm{rad}}$$

Force noise contribution

Sensing noise contribution

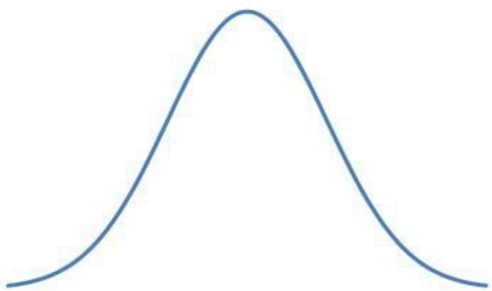
$$L_-^{(\mathrm{est})} = H_{\mathrm{A}} Y$$
$$= L_- + H_{\mathrm{A}}\delta F_{\mathrm{rad}} + \frac{1}{H_{\mathrm{IFO}}}\delta F_{\mathrm{rad}}$$

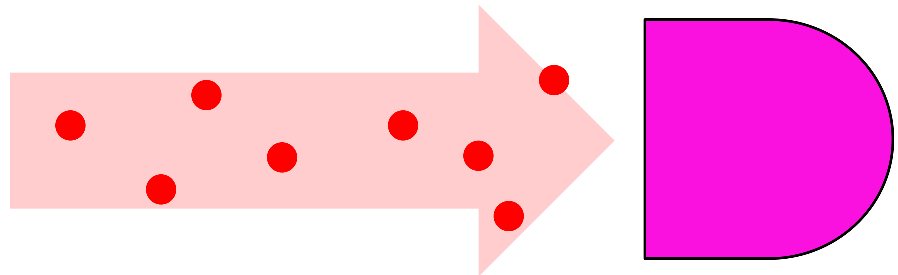
Estimated GW signal

Shot Noise

- Number of photons to photodiodes fluctuates

$$N \pm \sqrt{N}$$





Photodiode

Photon energy

$$p_1 = \frac{hc}{\lambda}$$

Number of photons

$$N = \frac{P_{\mathrm{PD}}}{p_1}$$

- Quantum fluctuation of power

$$\delta P_{\mathrm{shot}} = \sqrt{\frac{2hcP_{\mathrm{PD}}}{\eta\lambda}}$$

Shot noise spectrum

Quantum efficiency

Shot Noise Limit of Michelson

- Power change

$$H_{\mathrm{IFO}}(\omega) = \frac{\partial P_{\mathrm{PD}}}{\partial L_-} = \frac{2\pi P_0}{\lambda} \sin \frac{4\pi L_-}{\lambda}$$

- Shot noise

$$\delta P_{\mathrm{shot}} = \sqrt{\frac{2hcP_{\mathrm{PD}}}{\eta\lambda}} = \sqrt{\frac{hcP_0}{\eta\lambda}\left(1 - \cos\frac{4\pi L_-}{\lambda}\right)}$$

- Shot noise limited sensitivity

$$\delta L_{\mathrm{shot}} = \delta P_{\mathrm{shot}} \left(H_{\mathrm{IFO}}(\omega)\right)^{-1} \to \frac{1}{2\pi}\sqrt{\frac{hc\lambda}{2\eta P_0}}$$

$$\frac{\sqrt{1-\cos\phi}}{\sin\phi} = \frac{\sqrt{2\sin^2\frac{\phi}{2}}}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}} = \frac{1}{\sqrt{2}\cos\frac{\phi}{2}}$$

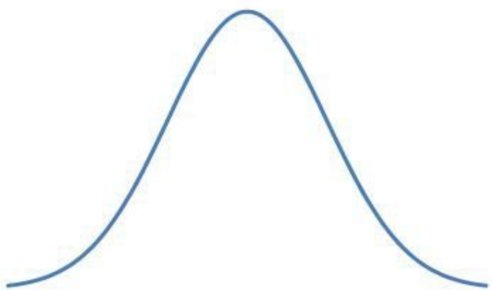
Better shot noise with higher input power

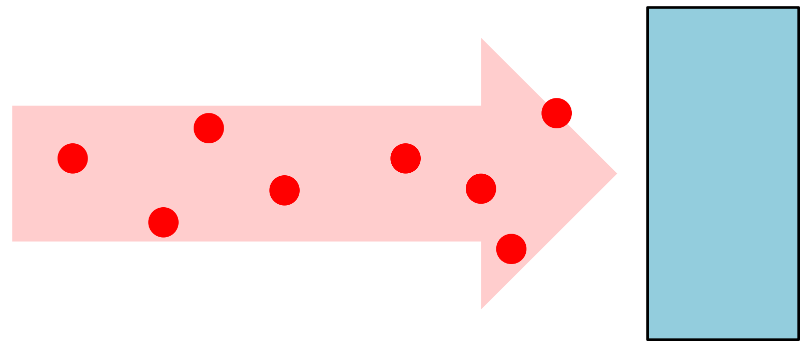
Best at dark fringe (where $P_{\mathrm{PD}}=0$)

Radiation Pressure Noise

- Number of photons to mirror fluctuates

$$N \pm \sqrt{N}$$





- Power fluctuation

Assumed Michelson with input power P_0

$$\delta P_{\mathrm{rad}} = \sqrt{\frac{hcP_0}{\lambda}}$$

- Mirror displacement

Force on mirror

Force to displacement (Actuator TF)

free-falling mirror (above resonant frequency)

$$\delta L_{\mathrm{rad}} = \frac{2\delta P_{\mathrm{rad}}}{c} H_{\mathrm{A}}(\omega) \simeq \sqrt{\frac{4hP_0}{c\lambda}} \frac{1}{m\omega^2}$$

Standard Quantum Limit

- Shot noise is lower with higher power

$$h_{\rm shot} = \frac{1}{2\pi L}\sqrt{\frac{hc\lambda}{2P_0}}$$

- Radiation pressure noise is lower with lower power

$$h_{\rm rad} = \frac{1}{m\omega^2 L}\sqrt{\frac{8hP_0}{c\lambda}}$$

Trade-off

√2 for two arms
m is mirror mass
(m/2 is reduced mass)
Assuming BS with infinite mass

- Standard Quantum Limit (SQL) for Michelson

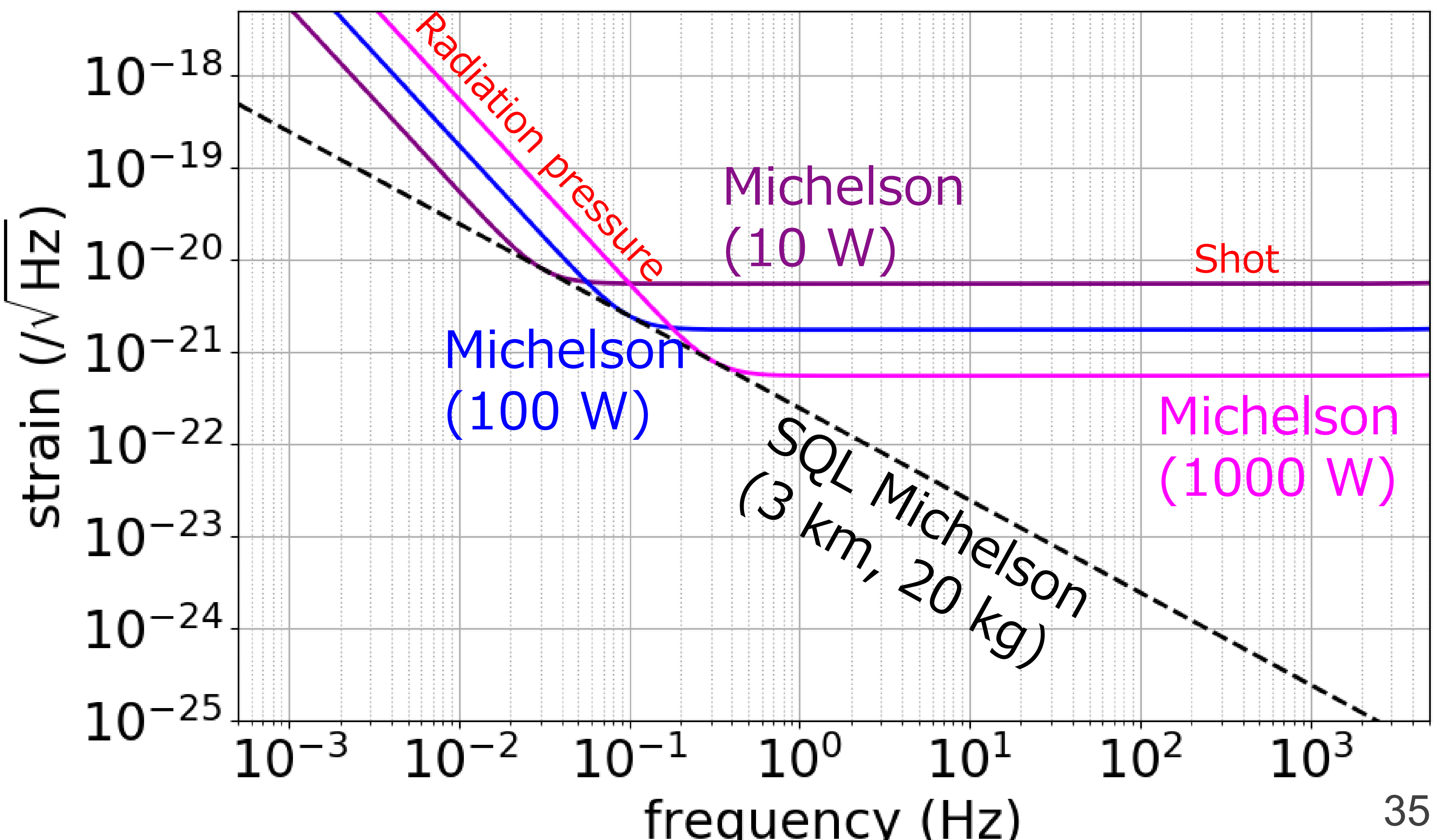
from Uncertainty principle

$$\hbar \equiv h/(2\pi)$$

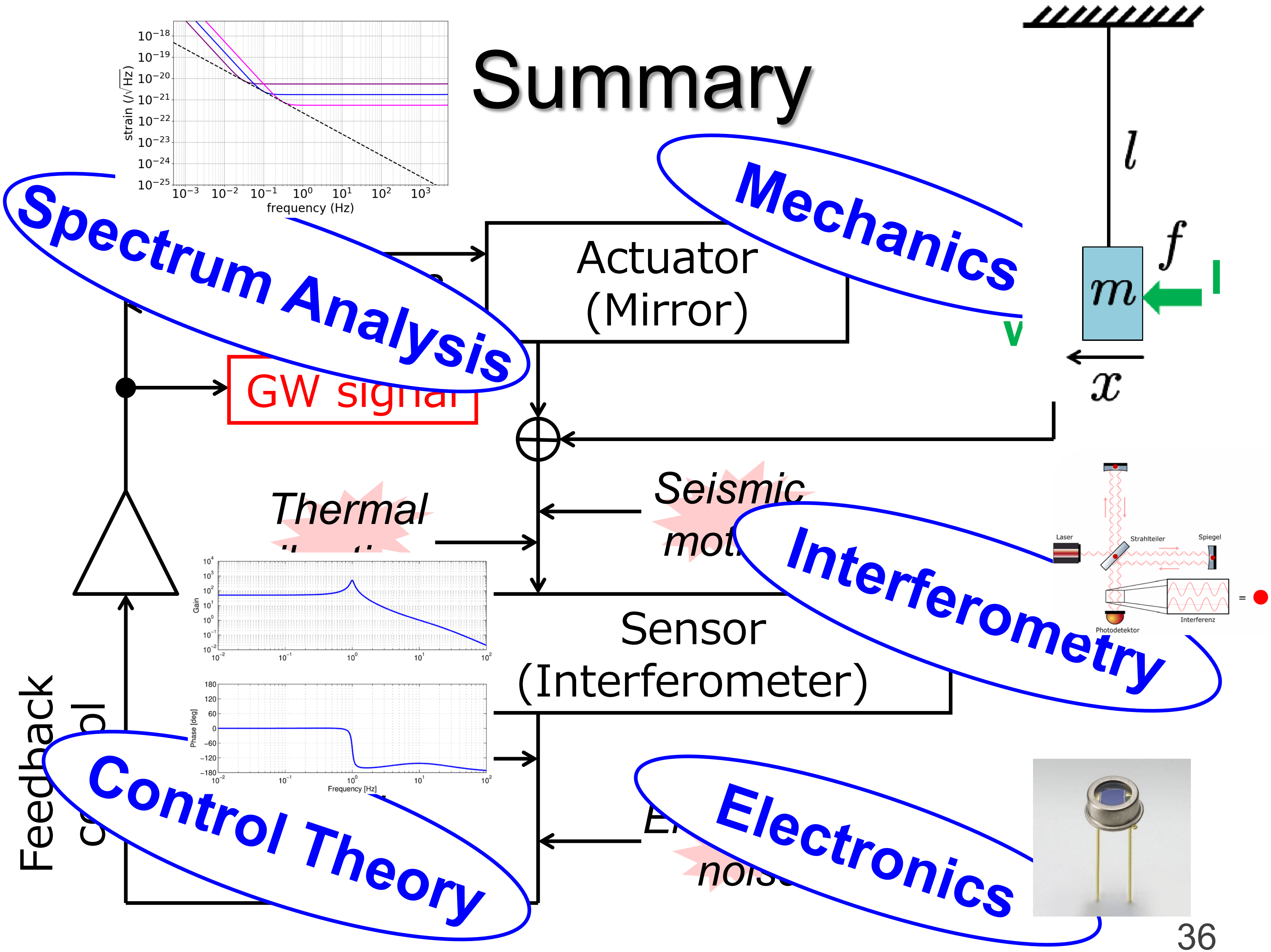
$$h_{\rm SQL} = \sqrt{2h_{\rm shot}h_{\rm rad}} = \sqrt{\frac{4\hbar}{m\omega^2 L^2}}$$

Input Power and Sensitivity

- SQL cannot be beaten by changing power



Summary



strain (/√Hz)
frequency (Hz)
Spectrum Analysis
Mechanics
Actuator
(Mirror)
l
f
m
x
GW signal
Thermal
Seismic
Interferometry
Laser
Strahlteiler
Spiegel
Photodetektor
Interferenz
Gain
Phase [deg]
Frequency [Hz]
Sensor
(Interferometer)
Feedback
Control Theory
Electronics