Optimal Sensor Fusion using $\mathcal{H}_\infty$ methods

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Introduction: Complementary Filters at KAGRA

- The pre-isolators of Type-A and Type-B suspensions in KAGRA are equipped with relative sensors and inertial sensors.
- A pair of complementary filters (low-pass and high-pass) can be used to combine the sensors into a virtual "super-sensor" that has superior noise properties.
- In this work, we discuss a scheme that uses $\mathcal{H}_\infty$ methods to synthesize complementary filters that optimally blend the sensors according to the sensor noises.

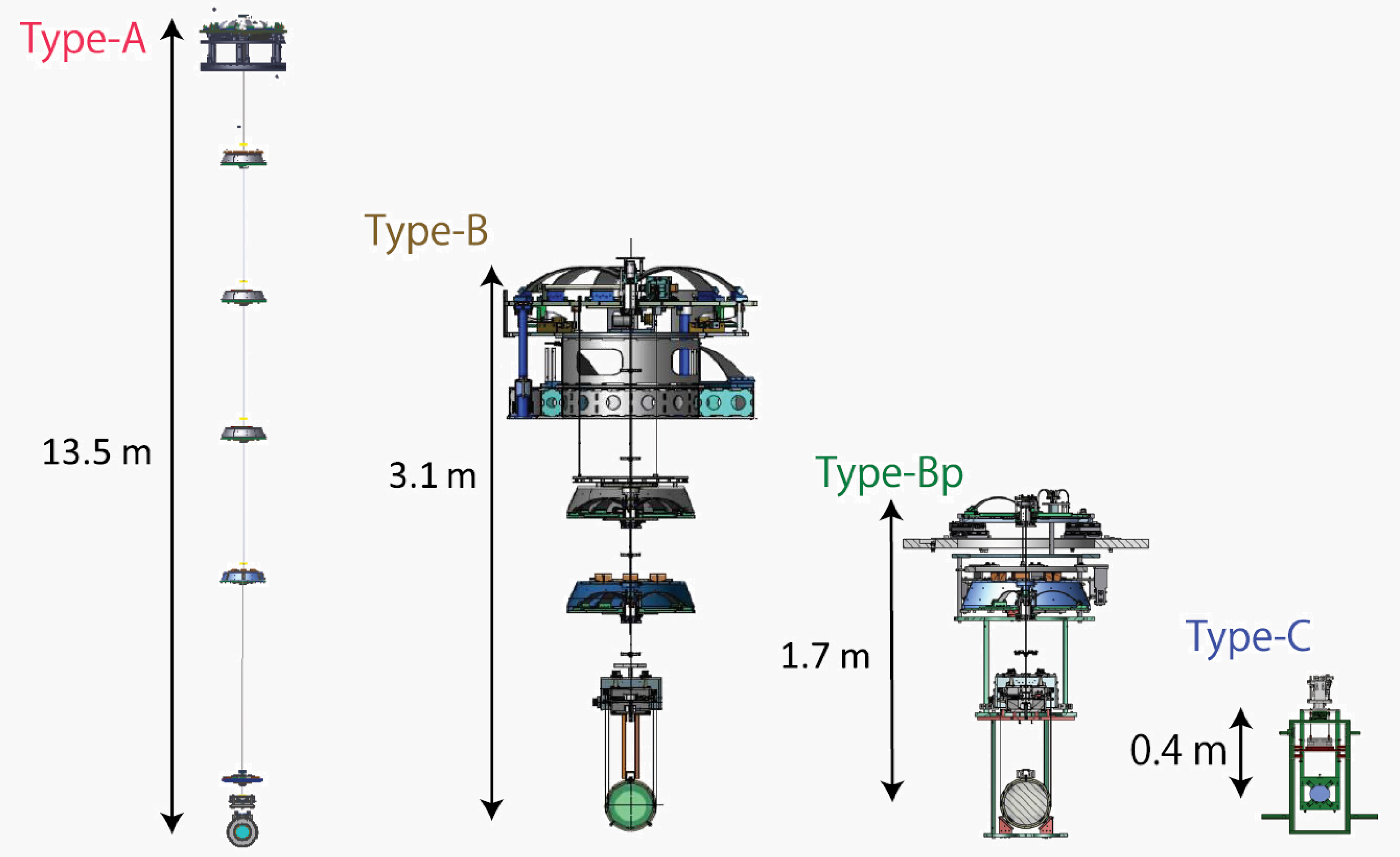

Figure 1: Type-A suspensions: input/end test masses, Type-B suspensions: beamsplitter and signal-recycling mirrors, Type-Bp suspensions: power-recycling mirrors, and Type-C suspensions: input/output mode cleaners [1].

Methodology: Complementary Filter Problem as an $\mathcal{H}_\infty$ Problem

$\mathcal{H}_\infty$ method in a nutshell:

1. As shown in Fig. 2, define signals w, z, u, and v, and hence,
2. derive a generalized plant $\mathbf{P}(s)$.
3. $\mathcal{H}_\infty$ synthesis gives an $\mathcal{H}_\infty$ optimal controller $\mathbf{K}_\infty(s)$ that minimizes the $\mathcal{H}_\infty$-norm of the closed-loop plant $\|\mathbf{G}(s;\mathbf{K},\mathbf{P})\|_\infty$.

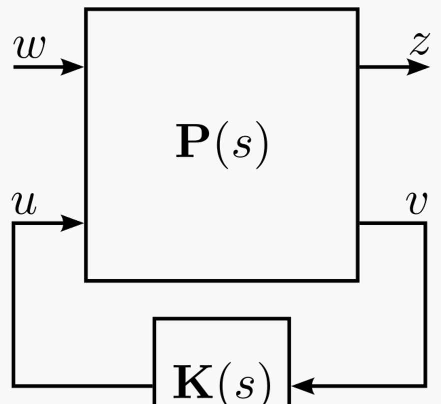

Figure 2: Generalized Plant Represenation.

- The close-loop transfer function $\mathbf{G}(s)$ is defined such that $z = \mathbf{G}(s)w$ and $u = \mathbf{K}(s)v$, i.e.

$$\mathbf{G}(s) = P_{11}(s) + P_{12}(s)\mathbf{K}(s)\left[\mathbf{I} - P_{22}(s)\mathbf{K}(s)\right]^{-1} P_{21}(s). \tag{1}$$

- The $\mathcal{H}_\infty$-norm is defined as

$$\|\mathbf{G}(s)\|_\infty = \sup_\omega \bar{\sigma}\left(\mathbf{G}(j\omega)\right), \tag{2}$$

where $\bar{\sigma}$ denotes the maximum singular value and ω is the angular frequency.

Methodology: Complementary Filter Problem as an $\mathcal{H}_\infty$ Problem

Formulating the complementary filter problem as an $\mathcal{H}_\infty$ problem:

Complementary filter configuration:
$N_1(s)$, $N_2(s)$: sensor noises,
$H_1(s)$, $H_2(s)$: complementary filters,
$N_\mathrm{super}(s)$: super sensor noise.

Step 1:
Apply constraint $H_1(s) + H_2(s) = 1$.

Step 2:
Model the sensor noises with transfer functions $\hat{N}_1(s)$ and $\hat{N}_2(s)$.

Φ_1 and Φ_2 are white noise processes that have unit intensity.

Step 3:
Add weighting functions $W_1(s)$ and $W_2(s)$.
Define the generalized plant $\mathbf{P}(\mathbf{s})$ as Eqn. (3).

$W_1(s)$ and $W_2(s)$ are the inverse of the frequency dependent specification of the sensor noises [2].

Finally:
$\mathcal{H}_\infty$ synthesis gives optimal $H_1(s)$, and get $H_2(s)$ from $1 - H_1(s)$.

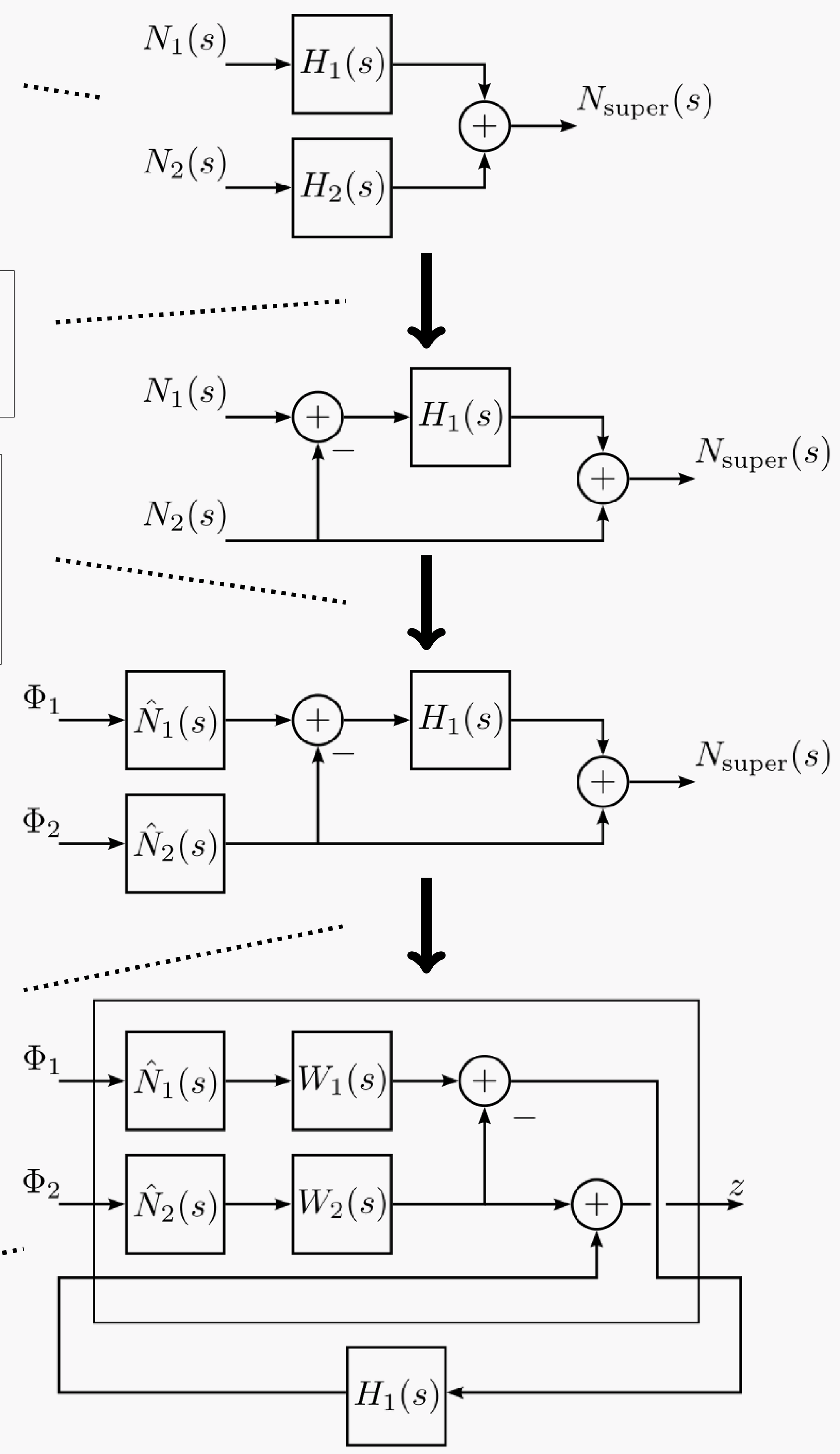

Figure 3: From a simple complementary filter configuration to generalized plant representation.

- The generalized plant is given by

$$\mathbf{P}(s) = \left[\begin{array}{cc|c} 0 & \hat{N}_2(s)W_2(s) & 1 \\ \hat{N}_1(s)W_1(s) & -\hat{N}_2(s)W_2(s) & 0 \end{array}\right]. \tag{3}$$

- The closed-loop plant is given by

$$\mathbf{G}(s) = \left[H_1(s)\hat{N}_1(s)W_1(s) \;\; H_2(s)\hat{N}_2(s)W_2(s)\right]. \tag{4}$$

- If we set $W_1(s) = 1/\hat{N}_2(s)$ and $W_2(s) = 1/\hat{N}_1(s)$, then the target specification of $N_1(s)$ is set to $N_2(s)$ when $|N_1(s)| \gg |N_2(s)|$, and vice versa.
- Minimizing $\|\mathbf{G}(s)\|_\infty$ gives optimal complementary filters that minimizes the maximum difference between the super sensor noise and the lower bound of the sensor noise in logarithmic scale.
- This is equivalent to minimizing the cost function

$$\boxed{J = \sup_\omega\left(\log|N_\mathrm{super}(j\omega)| - \log\min\left(|N_1(j\omega)|, |N_2(j\omega)|\right)\right).} \tag{5}$$

- $\mathcal{H}_\infty$ solvers are readily available in packages such as MATLAB and python-control.
- Complementary filters can be synthesized using the `kontrol` python package, which is developed for KAGRA's control systems [3].

Results: Complementary Filters for LVDTs and geophones

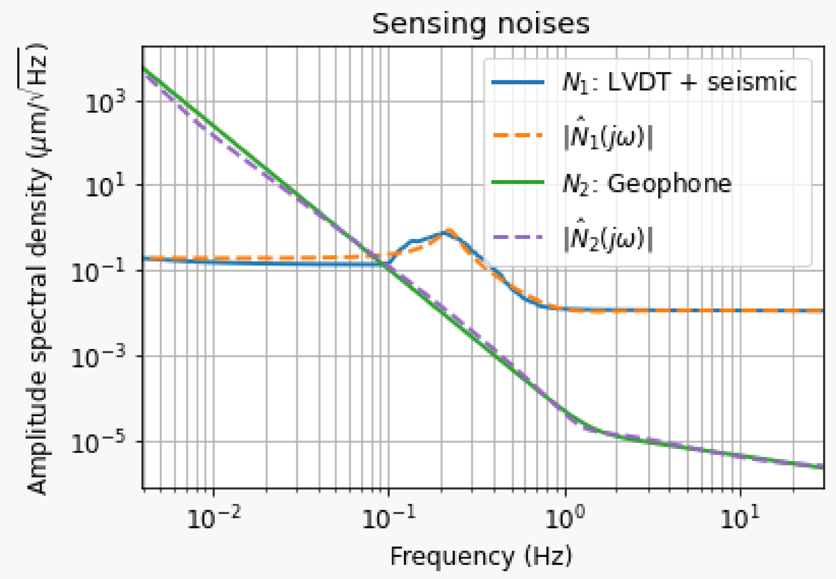

Figure 4: SRM sensor noises.

- Fig. 4: the empirical model of the seismic noise-coupled LVDT and geophone readout noises N_1 and N_2 and the transfer function models $\hat{N}_1(s)$ and $\hat{N}_2(s)$.
- Fig. 5: The complementary filters synthesized using $\mathcal{H}_\infty$ method.
- Fig. 6: The predicted super sensor noise using optimal complementary filters $H_1(s)$ and $H_2(s)$.

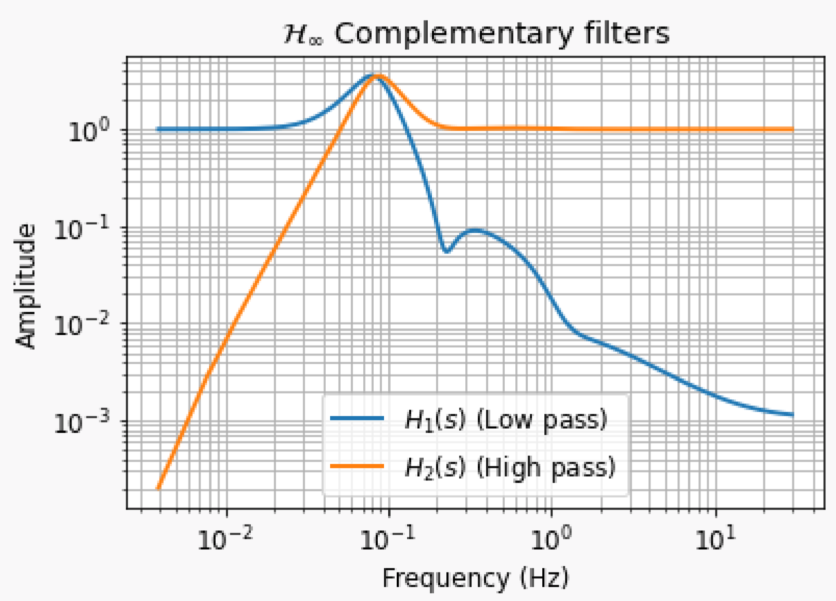

Figure 5: Filters (using proposed method)

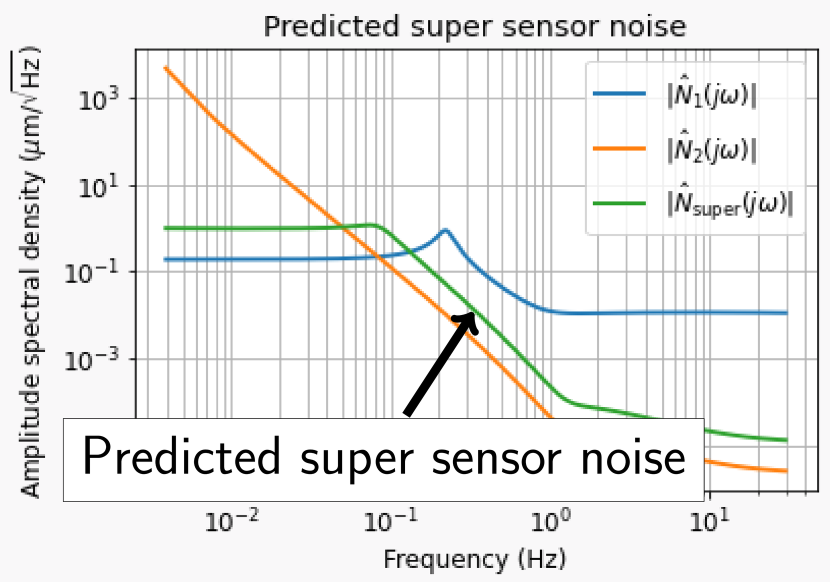

Figure 6: Predicted super sensor noise

- The predicted super sensor noise is defined by

$$|\hat{N}_\mathrm{super}(j\omega)| = \left[|H_1(s)|^2|\hat{N}_1(s)|^2 + |H_2(s)|^2|\hat{N}_2(s)|^2\right]^{\frac{1}{2}}. \tag{6}$$

- The super sensor noise is equally close to the lower bound at all frequencies (logarithmically).
- Preliminary implementation results can be found in Ref. [4].

Discussion and Future work.

- Unlike static filter designs, such us those in Refs. [5, 6, 7], the proposed method can generate filters that is optimal for any arbitrary sensor noises.
- Sensor correction and feedback-control filters can both be formulated into a complementary filter configuration. They can both be solved using the same method.

References

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