1 Optical levers

Conventionally, the optical lever (OL) conventionally refers to a device which can amplify small angular displacement. In KAGRA, optical levers are used for measuring the angular displacements (tilt) pitch P and yaw Y of the main optics. The optical lever beam path forms an angle of incidence with the optic axis of the main optic. For the OLs of Type-B suspensions, the incidence angle α on the vertical plane is non-zero while that on the horizontal plane is designed to be zero (See Fig. 1). Since the angle of incidence on the vertical plane is non-zero, a longitudinal shift of the optic will also cause the beam to be displaced in the same direction as caused by pitch of the optic, making the optic's longitudinal displacement also sensible. This displacement due to the longitudinal displacement of the optic is measured by another set of sensing optics which will be described in section 1.1 together with the tilt sensing optical lever setup. Note that term "optical lever" or "OL" under this context refers to the both setup as a whole but is not limited to the sensing of the angular displacement.

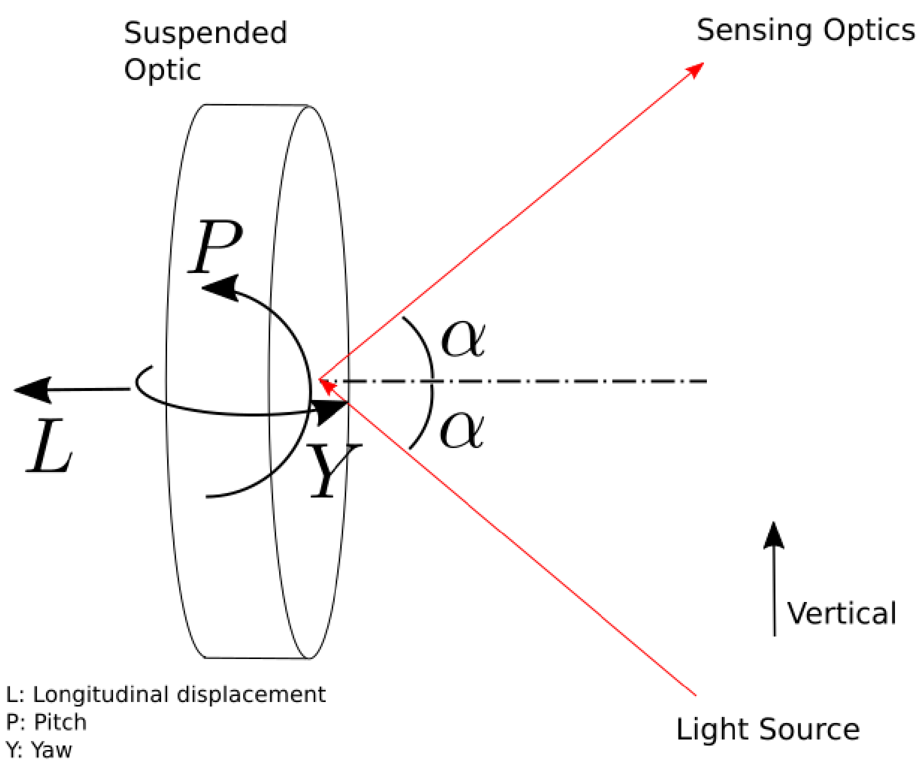

Figure 1: The angle of the incidence path of the Type-B suspension optical lever beam and the optic axis forms and an angle of incidence.

The optical lever plays an important role in interferometer commissioning. The sensor gives angular displacement of the optic thereby helping the restoration of a coarse alignment after maintenance or lock-loss. The optical lever also provides a reliable displacement and orientation readout of the optic that can be used to minimize the longitudinal to angle coupling via coil balancing and frequency dependent cancellation filters, further enhancing the stability of interferometer lock. Nevertheless, the optical lever signals are also used as error signal in the optic local control for damping of the longitudinal, pitch and yaw modes.

To fulfill the aforementioned purposes, it is expected the optical lever readouts are converted to longitudinal, pitch and yaw with minimal cross-couplings. The mathematical description relating the optic's displacement to the OL signals is derived and is discussed in section 1.2. A set procedures (diagonalization) based on the mathematics framework derived is also followed to ensure that the signals are reasonably accurate and decoupled. This is discussed in later sections.

1.1 Optical lever setup

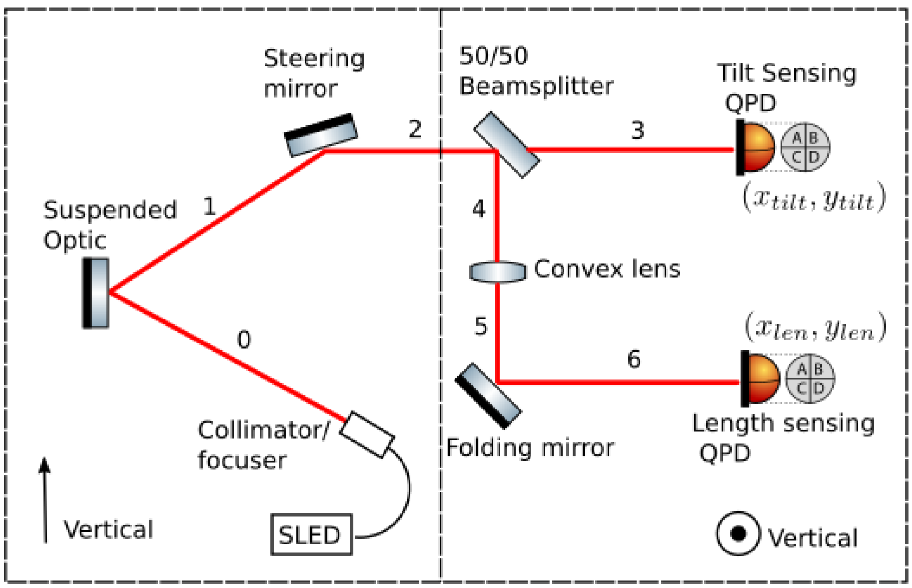

Figure 2: Optical lever layout of the Type-B suspension. Left: Elevation view. Right: Plan view.

The optical lever setup consists of superluminescent light-emitting diode (SLED), a collimator/focuser, the optic itself, a steering mirror, a 50/50 beamsplitter, a convex lens, another folding mirror, and two quadrant photodetectors (QPD), see Fig. 2. The QPDs are light beam position sensors each consist of four photodiodes placed on different quadrants. The horizontal and vertical displacements of the light beam at the QPD plane can be derived from the amount of light that is received by each photodiode. Theoretically, each of the QPDs might be sensitive to both tilt and length. But, we refer the one that is part of the actual angular sensing OL "Tilt-sensing QPD" and the other "Length-sensing QPD" under this context. The light source and the collimator/focuser are placed on an optical table outside the vacuum chamber of the suspended optic. The beam begins from a position below the optic and travels at an angle pointing upward. The beam then enters the vacuum chamber via a lower-viewport. After reflecting off the optic, the beam reaches a steering mirror located at the edge of another optical table which is placed at a position outside the chamber and higher than the optic. The beam path and the optic axis forms an incidence angle α (typically around 37°). The steering mirror redirects the beam to follow a horizontal path parallel to the optical table. The beam will first meet a beamsplitter which will split the beam into two beams. One beam will go straightforwardly to the tilt-sensing QPD, while the other will reach the length-sensing QPD after going through a convex lens with focal length of f (typically 300 mm). And, not so importantly, the purpose of the folding mirror is simply folding the beam path such that all components can fit onto a small optical table.

1.2 Optical lever QPD readout

Each of the QPDs is calibrated to give the horizontal and vertical positions of the beamspot casting on it. The horiztonal and vertical displacements of the beamspot at the tilt-sensing QPD plane are denoted as x_{tilt} and y_{tilt} respectively, while that at the length-sensing QPD plane are denoted as x_{len} and y_{len} respectively.

If the components are aligned perfectly, the relationship between the the optical lever readouts x_{tilt}, y_{tilt}, x_{len} and

y_{len}, and the optic's displacement L, P and Y can be easily deduced geometrically, as shown in Fig. 3.

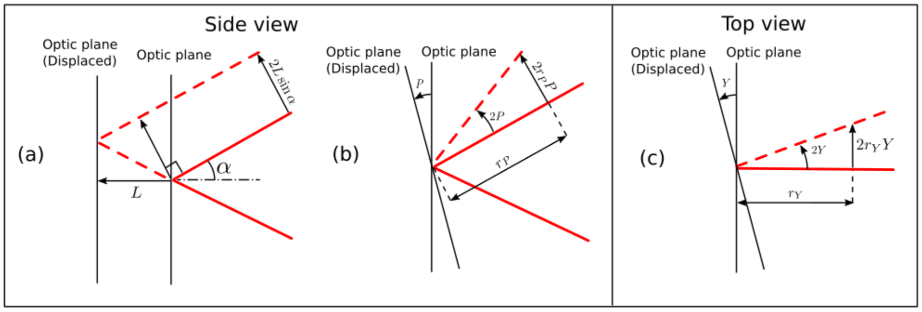

Figure 3: The displaced optic caused the optic lever beam (red solid line) to be displaced (dashed red line)

Fig. 3 (a) and (b) show the side view of the optical lever beam path where the effect of the longitudinal and pitch displacement on the light beam is visible; (c) shows the top view of the beam path where the effect of yaw on the OL beam is more clear. Since the steering mirror bends the light beam follow a general horizontal path, the displacements of the beam in (a) and (b) are rendered as vertical displacement to the tilt-sensing QPD, whereas that in (c) is shown as horizontal displacement.

As can be seen in Fig. 3 (a),The longitudinal displacement L of the optic caused the incidence beam travel an additional distance of $L/\cos\alpha$. As a result, the reflected beam compared to the orignal one is shifted by $2L\sin\alpha$. From (b), the angular displacement pitch P of the optic redirects the reflected beam to travel at an angle $2P$, that lies on the vertical plane, from the original beam path and causes a beam displacement of $2r_PP$ in the same direction as (a) at distance r_P away from the optic. Similarly, as shown in (c), the yaw angular displacement of the optic caused the reflected beam to be deflected at an angle of $2Y$ on the horizontal plane relative to the original reflection path and causes a horizontal beam displacement of $2r_YY$ at distance r_Y away from the optic. Note that the quantity r_P and r_Y are referred to the lever arms which amplify the pitch and yaw angular displacements. The two lever arms are not independent to each other by is related by the angle of incidences α by $r_Y = r_P\cos\alpha$. So, the horizontal beamspot displacement x_{tilt} and the vertical beamspot displacement y_{tilt} at the tilt QPD plane relate to the optic's displacement (L, P, Y) as

$$x_{\text{tilt}} = (2r_Y)\,Y, \tag{1}$$

and

$$y_{\text{tilt}} = (2r_P)\,P + (2\sin\alpha)\,L, \tag{2}$$

As for the length-sensing QPD, let r_L be the distance between the optic and the convex lens (path segments 1-2-4 in Fig. 2) and d be the distance between the convex lens (path segments 5-6 in Fig. 2). From the ray transfer matrix, the beam spot vertical displacement y_{len} and the angle between the beam and optical lever axis γ at the the length-sensing QPD plane gives

$$\begin{pmatrix} y_{\text{len}} \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & r_L \\ 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} (2\sin\alpha)\,L \\ 2P \end{bmatrix}. \tag{3}$$

Simplifying Eqn. 3, gives the position of the beamspot at the length-sensing QPD plane:

$$\begin{aligned} y_{\text{len}} &= \begin{bmatrix} 1-\frac{d}{f} & \left(1-\frac{d}{f}\right) r_L + d \end{bmatrix} \cdot \begin{bmatrix} (2\sin\alpha)\,L \\ 2P \end{bmatrix} \\ &= \left(1-\frac{d}{f}\right)(2\sin\alpha)\,L + 2\left[\left(1-\frac{d}{f}\right) r_L + d\right] P. \end{aligned} \tag{4}$$

By placing the length-sensing QPD exactly at the image plane $d = \frac{r_L f}{r_L - f}$, the coefficient of P is set to become zero, i.e. $\left(1-\frac{d}{f}\right) r_L + d = 0$, making y_{len} only sensitive to L but not P. And, the vertical displacement of the beamspot at the length-sensing QPD plane becomes

$$y_{\text{len}} = \left(\frac{-2f\sin\alpha}{r_L - f}\right) L. \tag{5}$$

Equation 1, 2 and 5 provides three equations relating the the dynamics of the optics (L, P, Y) to the three sensor outputs (x_{tilt}, y_{tilt}, y_{len}). Therefore, there are enough information for us to extract the position of the optic. Using the information given here, the coupling ratios between all degrees of freedom can, in principle, be reduced to 0. But, in practice, the coupling ratios will still be usually at around 0.1 due to other sources of cross-coupling which will be discussed in section 1.3.

1.3 Optical lever additional cross-couplings

Equations in section 1.2 assumes perfect alignment and good measurement of various length parameters such as the effective lever arms r_Y and r_P. In ideal case, x_{tilt} is assumed to be only sensitive to yaw, y_{tilt} is assumed to be generally sensitive to pitch, while y_{len} is assumed to be a pure measurement of length. However, in reality, the alignment and the measurements are not flawless and all degrees of freedom will be cross-coupled to each other. Pitch and yaw will be cross-coupled and will be seen in both tilt-sensing QPD readouts x_{tilt} and y_{tilt} (Pitch to yaw and yaw to pitch cross-couplings). Also, as in seen in Eqn. 2 already, longitudinal to tilt coupling is caused by a non-zero angle of incidence and will generally make x_{tilt} and y_{tilt} also a function of the optic's longitudinal displacement. Lastly, the length-sensing QPD will also measure some fraction pitch and yaw motion because the optical lever beam is not centered at the optic or due to misplacement of the length-sensing QPD. All these mechanisms of cross-coupling should be taken into account and the mathematical relationship between the QPD readouts/beamspot displacements (x_{tilt}, y_{tilt}, x_{len}, y_{len}) and the optic's displacements (L, P, Y) should be modified from Eqn. 1, 2 and 5 accordingly.

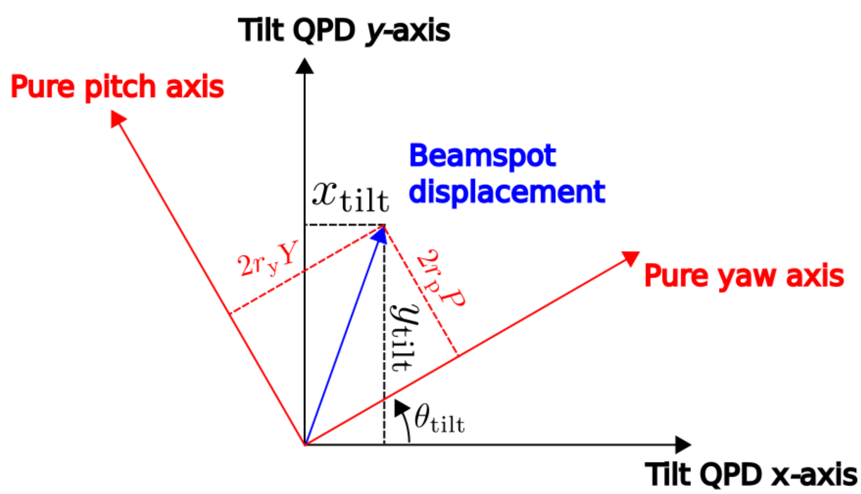

Figure 4: The tilt frame (red) appears to be rotated compared to the tilt-sensing QPD's frame of reference (black). So, x_{tilt} and y_{tilt} reads differently from Eqn. 1 and 2 if the tilt of the optic caused the beamspot to displaced (blue).

Yaw to pitch and pitch to yaw cross-coupling could be introduced by a rotation frame of reference defined by pitch and yaw at the QPD plane (tilt frame) and the tilt-sensing QPD's frame of reference, see Fig. 4. In the figure, the tilt frame of reference refers to the orthogonal axes (shown in red) that align with the beamspot displacement when the optic tilts in pure pitch and yaw respectively. This misalignment can be introduced by a misaligned steering mirror which rotates the plane of incidence of the optical lever. In addition, the QPD itself might also be tilted w.r.t the horizontal due to design tolerances or other factors. Therefore, both x_{tilt} and y_{tilt} of the tilt QPD readouts will measure a mixture of pitch and yaw. Mathematically saying, the R.H.S of the Eqn. 1 and 2 froms a vector $[(2r_Y)\,Y,\,(2r_P)\,P + (2\sin\alpha)\,L]^{\mathrm{T}}$ and this vector is rotated by an angle of θ_{tilt} which is the angle between the tilt-sensing QPD x-axis and the pure yaw axis, as shown in Fig. 4. Applying the rotational matrix gives a new set of equations for the tilt-sensing readouts:

$$\begin{pmatrix} x_{\text{tilt}} \\ y_{\text{tilt}} \end{pmatrix} = \mathbf{R}\left(\theta_{\text{tilt}}\right) \cdot \begin{bmatrix} (2r_Y)\,Y \\ (2r_P)\,P + (2\sin\alpha)\,L \end{bmatrix}, \tag{6}$$

where

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is the usual 2-D rotational matrix.

Similarly, if the optic displaces in longitudinal direction, the beamspot displacement at the length-sensing QPD plane does not necessarily align to y_{len} and would require a similar modification to Eqn. 5. Therefore, the new equations for the length-sensing QPD becomes

$$\begin{pmatrix} x_{\text{len}} \\ y_{\text{len}} \end{pmatrix} = \mathbf{R}(\theta_{\text{len}}) \cdot \begin{bmatrix} 0 \\ \left(\frac{-2f\sin\alpha}{r_L - f}\right) L \end{bmatrix}. \tag{7}$$

Moreover, besides the angle of incidence α on the vertical plane, there will inevitably be a angle of incidence β on the horizontal plane. This will cause the reflection plane to be misaligned with the incidence plane and create length to yaw coupling like the way pitch readout is coupled to longitudinal displacement L, see Fig. 3. Therefore, a $(2\sin\beta)\,L$ horizontal shift should be added to the first row of the R.H.S vector in Eqn. 6 to take into account for this type of coupling. So Eqn. 6 becomes

$$\begin{pmatrix} x_{\text{tilt}} \\ y_{\text{tilt}} \end{pmatrix} = \mathbf{R}(\theta_{\text{tilt}}) \cdot \begin{bmatrix} (2r_Y)\,Y + (2\sin\beta)\,L \\ (2r_P)\,P + (2\sin\alpha)\,L \end{bmatrix}. \tag{8}$$

In addition, the horizontal shift of the beam due to longitudinal displacement will also be sensible by the length-sensing QPD. Therefore, the first row of Eqn. 7 should also be modified, i.e.

$$\begin{pmatrix} x_{\text{len}} \\ y_{\text{len}} \end{pmatrix} = \mathbf{R}(\theta_{\text{len}}) \cdot \begin{bmatrix} \left(\frac{-2f\sin\beta}{r_L - f}\right) L \\ \left(\frac{-2f\sin\alpha}{r_L - f}\right) L \end{bmatrix}. \tag{9}$$

Nevertheless, tilt to length coupling can arise from an off-centered optical lever beam. Instead of actual longitudinal displacement L, the length displacement that is sensed by the QPDs as shown in Eqn. 7 and 8 is the displacement of the beamspot L_{beam} at the suspended optic. So, the equations should be modified to

$$\begin{pmatrix} x_{\text{tilt}} \\ y_{\text{tilt}} \end{pmatrix} = \mathbf{R}(\theta_{\text{tilt}}) \cdot \begin{bmatrix} (2r_Y)\,Y + (2\sin\beta)\,L_{\text{beam}} \\ (2r_P)\,P + (2\sin\alpha)\,L_{\text{beam}} \end{bmatrix}, \tag{10}$$

and

$$\begin{pmatrix} x_{\text{len}} \\ y_{\text{len}} \end{pmatrix} = \mathbf{R}(\theta_{\text{len}}) \cdot \begin{bmatrix} \left(\frac{-2f\sin\beta}{r_L - f}\right) L_{\text{beam}} \\ \left(\frac{-2f\sin\alpha}{r_L - f}\right) L_{\text{beam}} \end{bmatrix}. \tag{11}$$

Normally, L and L_{beam} are synchronized. However, if the optical lever beamspot is offset from the optical axis of the optic, the beamspot displacement L_{beam} will not be the same as the longitudinal displacement of the optic L but will gain additional offsets of $\delta_Y Y$ and $\delta_P P$, i.e.

$$L_{\text{beam}} = L + \delta_P P + \delta_Y Y, \tag{12}$$

where δ_Y and δ_P are the beamspot's horizontal and vertical offsets at the suspended optic. Combining Eqn. 10, 11 and 12, the ultimate equation relating the QPDs readout to the optic's displacement is

$$\begin{gathered} \begin{pmatrix} \cos\theta_{\text{tilt}} & \sin\theta_{\text{tilt}} & 0 & 0 \\ -\sin\theta_{\text{tilt}} & \cos\theta_{\text{tilt}} & 0 & 0 \\ 0 & 0 & -\sin\theta_{\text{len}} & \cos\theta_{\text{len}} \end{pmatrix} \begin{pmatrix} x_{\text{tilt}} \\ y_{\text{tilt}} \\ x_{\text{len}} \\ y_{\text{len}} \end{pmatrix} = \\ \begin{pmatrix} 2\sin\beta & 2\delta_P\sin\beta & 2r_Y + 2\delta_Y\sin\beta \\ 2\sin\alpha & 2r_P + 2\delta_P\sin\alpha & 2\delta_Y\sin\alpha \\ \frac{-2f\sin\alpha}{r_L - f} & \frac{-2\delta_P f\sin\alpha}{r_L - f} & \frac{-2\delta_Y f\sin\alpha}{r_L - f} \end{pmatrix} \begin{pmatrix} L \\ P \\ Y \end{pmatrix}. \end{gathered} \tag{13}$$

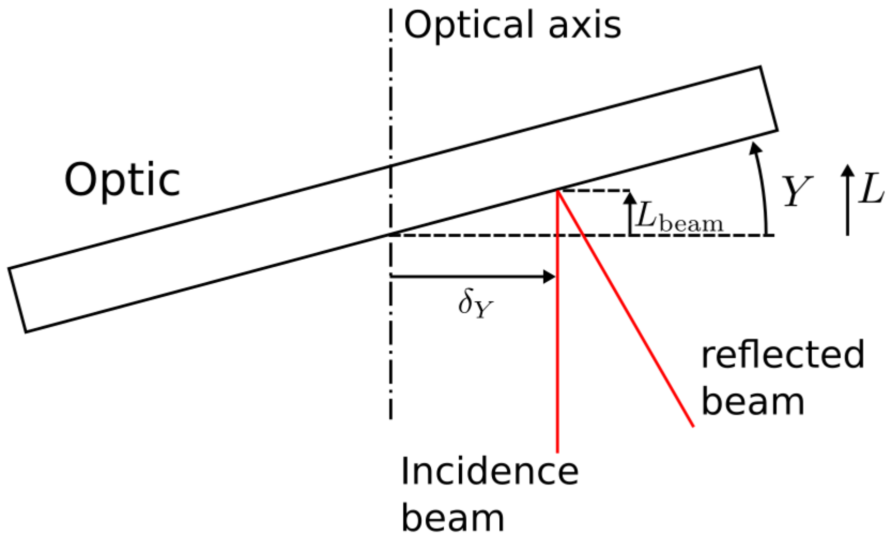

Figure 5: An addition of beam displacement of $\delta_Y Y$ in the longitudinal direction will be introduced by an optical lever beam off-centered by δY horizontally when the optic is tilted in yaw direction by Y.

Note that the first equation of Eqn. 11 is omitted because of redundancy.

Besides the off-centered beam, tilt to longitudinal could also be caused by misplacement of the length-sensing QPD. The derivation for converting the QPD readouts to longitudinal displacement, starting from Eqn. 5, assumes that the length-sensing QPD is placed exactly at $d = \frac{r_L f}{r_L - f}$ from the convex lens. In reality, it is unlikely that the placement of the QPD is perfect. Therefore, the length-sensing QPD should measure a fraction of the tilt motion regardless of the position of the beamspot at the main optic. Since the correction required for this type of coupling is similar to that of compensating the couplings resulting from the off-centered beam, it is possible to interpret all tilt to longitudinal couplings as one type and use only δ_P and δ_Y for to account for both two types of couplings.

1.4 Optical lever diagonalization

To obtain the signals (L, P, Y) from the QPD signals (x_{tilt}, y_{tilt}, x_{len}, y_{len}), it is necessary to obtain the parameters written in Eqn. 13 in order to diagonalize the signals. A list of the parameters and descriptions is shown (table 1). The tactics for obtaining these numbers are discussed after.

For the optical lengths r_L, r_P and r_Y, and the incidence angle on the vertical plane α, they are more or less fixed by the design. These can be easily obtained by looking up the in the CAD drawings and by physically measuring distances between the various optic components on the optical table.

As for the rotational angles θ_{tilt} and θ_{len}, beamspot offsets δ_Y and δ_P, and the angle of incidence β, it is not easy to directly measure them and will require inferring the coupling ratios in the displacement amplitude spectrum in order to indirectly obtain them.

The idea of using amplitude spectrum utilizes the fact that longitudinal, pitch and yaw modes are mostly distinguishable in frequency space. When the optic resonates, the optic only oscillates in a certain way. This means that the contribution to the displacement spectrum at a particular resonance frequency is purely from the corresponding mode. If there exists modes that involve only 1 of the 3 directions (L, P, Y), the Eqn. 13 can be simplified drastically at the corresponding frequencies because the other 2 displacements can be nullified. From the SUMCON model (see section **??**), a pure pitch mode can be identified at around 0.85 Hz, whereas two pure yaw modes can be found at around 1 Hz and 1.38 Hz. Although a pure longituidal mode does not exist, the main pendulum modes at around 0.65 Hz involving both L and P serves as a good reference for identifying the longitudinal motion. Moreover, the longitudinal displacement at the IP stage is assumed to be synchronized with the optic's longitudinal displacement

	Parameters	Description
1	r_L	The length of the OL path from the optic to the lens.
2	r_P	The effective beam length amplifying pitch.
3	r_Y	The effective beam length amplifying yaw.
4	θ_{tilt}	Misalignment angle of the tilt QPD frame. (Fig. 4)
5	θ_{len}	Misalignment angle of the length QPD frame. (Eqn. 11)
6	α	Angle of incidence of the OL on the vertical plane
7	β	Angle of incidence of the OL on the horizontal plane
8	δ_Y	Horizontal (transverse) offset of the OL beamspot at the optic.
9	δ_P	Vertical offset of the OL beamspot at the optic.

Table 1: Parameters relating the QPD readouts and the optic displacement as shown in Eqn. 13

but not any pitch at DC. Thus, an offset in the IP longitudinal direction can be used to further decouple L and P if necessary.

Before measurements are done, the length sensing QPD has to be placed exactly at $d = \frac{r_L f}{r_L - f}$ from the convex lens in order for the equations to be valid (see Eqn. 4). Since r_L is directly measurable, the QPD can be placed approximately at the right position. But, this is usually not enough and further fine tuning had to be done. Originally, the length sensing QPD is designed to only sense longitudinal displacement. Therefore, the position of the length sensing QPD was tuned such that it is coupled to minimum tilt. This is done with an iterative process. The first step is to excite the pure pitch mode at 0.85 Hz and measure the length sensing QPD readouts. Then the position of the QPD was adjusted with a fine transnational stage until the a minimal 0.85 Hz signal is reached. This does not guarantee that the QPD is placed accurately at the position but a reasonable good estimation assuming minimal coupling due to the OL beam offset δ_P. In doing so, the coupling from pitch is minimized and the L signal is maximized, validating Eqn. 5 as a reasonably good equation to begin with. Any further couplings can be canceled later by introducing the parameters in table 1.

The first two parameters that can be obtained from measurements are θ_{len} and θ_{tilt}. As is shown Fig. 4, this is because they only depend on the ratios between the measured vertical and horizontal readouts from the QPDs but not the actual longitudinal, pitch or yaw displacement. To obtain θ_{len}, the 0.65 Hz longitudinal mode was first stimulated. Then, the horizontal and vertical beamspot displacement at the length-sensing QPD plane at that frequency, $x_{\text{len}}|_{f=0.65\,\text{Hz}}$ and $y_{\text{len}}|_{f=0.65\,\text{Hz}}$, can be used to calculate the rotational angle θ_{len} as follows. Since pitch and yaw motion are very small compared to longitudinal motion, Eqn. 13 can be simplified to Eqn. 9 and can be further simplified to Eqn. 7 if β is small, which is the case for Type-B suspensions. From Eqn. 7, it can be easily seen that

$$\theta_{\text{len}} = \arctan\left(-\frac{x_{\text{len}}}{y_{\text{len}}}\right)\bigg|_{f=0.65\,\text{Hz}}. \tag{14}$$

By applying this angle θ_{len}, the length-sensing QPD frame can be aligned very close to the desired frame and the equation

$$L \approx -\frac{r_L - f}{2f\sin\alpha}\begin{pmatrix}-\sin\theta_{\text{len}} & \cos\theta_{\text{len}}\end{pmatrix}\begin{pmatrix}x_{\text{len}}\\ y_{\text{len}}\end{pmatrix}, \tag{15}$$

derived from Eqn. 7, now gives a very good approximated of the longitudinal displacement of the optic, at least at the longitudinal resonance frequency. This can be cross-verified by offsetting the longitudinal displacement at the IP stage and comparing the two displacements measured by the IP LVDTs and Eqn. 15.

Similarly, the same approach can be used for obtaining θ_{tilt}. In this case, exciting the pitch mode at 0.85 Hz or the yaw modes at 1 Hz and 1.38 Hz will simplify Eqn. 13 at those frequencies to either

$$\begin{pmatrix} x_{\text{tilt}} \\ y_{\text{tilt}} \end{pmatrix} = \mathbf{R}\left(\theta_{\text{tilt}}\right) \begin{bmatrix} 0 \\ \left(2r_P\right) P \end{bmatrix}, \tag{16}$$

if x_{tilt} and y_{tilt} are evaluated at the pitch resonance frequency (0.85 Hz) while the pitch mode was excited, or

$$\begin{pmatrix} x_{\text{tilt}} \\ y_{\text{tilt}} \end{pmatrix} = \mathbf{R}\left(\theta_{\text{tilt}}\right) \begin{bmatrix} \left(2r_Y\right) Y \\ 0 \end{bmatrix}, \tag{17}$$

if x_{tilt} and y_{tilt} are evaluated at the yaw resonance frequency (1 Hz, 1.38 Hz) while the yaw modes were stimulated. For the former case, the angle θ_{tilt} can be derived as

$$\theta_{\text{tilt}} = \arctan\left(-\frac{x_{\text{tilt}}}{y_{\text{tilt}}}\right)\Bigg|_{f=0.85\,\text{Hz}}. \tag{18}$$

As for the latter case, the angle θ_{tilt} can be calculated as

$$\theta_{\text{tilt}} = \arctan\left(\frac{y_{\text{tilt}}}{x_{\text{tilt}}}\right)\Bigg|_{f=1\,\text{Hz or }1.38\,\text{Hz}}. \tag{19}$$

In all cases, the obtained θ_{tilt}s should be very similar, if not the same. One of the reasons why θ_{tilt} can be different when using Eqn. 18 and 19 is because the optical lever beamspot displacements at the optic, δ_P and δ_Y, were assumed to be zero, but, in fact, there could be non-zero but very small. From Eqn. 10, δ_P and δ_Y in L_{beam} couple to $\left(2r_P\right) P$ more strongly compared to $\left(2r_Y\right) Y$ because α is much greater than β by design. In this sense, Eqn. 16 is less accurate than Eqn. 17. Therefore, in the case if Eqn. 18 and Eqn. 19 give different θ_{tilt}, the one from calculated using the latter equation should be used. In any case, the coupling ratios between pitch to yaw and yaw to pitch should be very small (typically lower than 0.01 μrad/μrad) regardless of which θ_{tilt} is chosen. From here, estimation of pitch and yaw at their corresponding resonance frequencies can be done by applying θ_{tilt} to Eqn. 6 where $L \approx 0$ when the optic is not excited at those frequencies in longitudinal DoF.

After obtaining θ_{tilt} and θ_{len}, the rest of the parameters β, δ_Y and δ_P can be obtained by applying the known parameters to Eqn. 10 and Eqn. 12. For β, it can be obtained by stimulating the longitudinal mode and measuring at the corresponding frequency. With only the longitudinal mode (0.65 Hz) excited, the first equation at 0.65 Hz from Eqn. 10 can be simplified to obtain β, i.e.

$$\beta = \arcsin\left(\frac{x_{\text{tilt}}\cos\theta_{\text{tilt}} + y_{\text{tilt}}\sin\theta_{\text{tilt}}}{2L}\right)\Bigg|_{f=0.65\,\text{Hz}}. \tag{20}$$

As for the beam position offsets δ_Y and δ_P, they can be obtained by separately excitation and measurement of yaw mode and pitch mode respectively. This can be easily seen from Eqn. 12. When only yaw mode is excited, at the yaw resonance frequencies, the equation can be rearranged to

$$\delta_Y = \frac{L_{\text{beam}}}{Y}\Bigg|_{f=1\,\text{Hz or }1.38\,\text{Hz}}, \tag{21}$$

while in the case of pitch, the equation can be rearranged to

$$\delta_P = \frac{L_{\text{beam}}}{P}\Bigg|_{f=0.85\,\text{Hz}}. \tag{22}$$

In both cases, it is required to measure L_{beam} and the corresponding angular displacement, pitch or yaw. Eqn. 15 was said to be an approximation of longitudinal displacement because δ_Y and δ_P were set to zero. And, in fact, the left handside of the equation should be replaced by L_{beam} so the equation is exact, i.e.

$$L_{\text{beam}} = -\frac{r_L - f}{2f\sin\alpha}\begin{pmatrix} -\sin\theta_{\text{len}} & \cos\theta_{\text{len}} \end{pmatrix}\begin{pmatrix} x_{\text{len}} \\ y_{\text{len}} \end{pmatrix}. \tag{23}$$

This equation can be derived from Eqn. 11 as well but this was not written this way in the first place to avoid confusion. Meanwhile, assuming that $2r_Y \gg 2\delta_Y \sin\beta$ and $2r_P \gg 2\delta_P \sin\alpha$ ($2r_Y$ and $2r_P$ are in the order of 10^0 m while δ_Y and δ_P are typically in the order of 10^{-2} m or less), yaw and pitch can be estimated by

$$\begin{pmatrix} Y \\ P \end{pmatrix} = \begin{pmatrix} \frac{1}{2r_Y} & 0 \\ 0 & \frac{1}{2r_P} \end{pmatrix} \begin{pmatrix} \cos\theta_{\mathrm{tilt}} & \sin\theta_{\mathrm{tilt}} \\ -\sin\theta_{\mathrm{tilt}} & \cos\theta_{\mathrm{tilt}} \end{pmatrix} \begin{pmatrix} x_{\mathrm{tilt}} \\ y_{\mathrm{tilt}} \end{pmatrix}, \tag{24}$$

which can be seen from Eqn. 10. So, to obtain the parameters δ_Y and δ_P, yaw modes and pitch modes were excited and measured separately. Then, L_{beam}, P and Y can be calculated using Eqn. 23 and 24. Lastly, the values can be substituted into Eqn. 21 and 22 to obtain the optical lever beam offset at the optic plane.

As can be seen, the normal modes provide very convenient directional reference of the suspension. Concentrating in resonance peaks in the displacement spectrum of the QPD readout while modes are excited simplifies the gigantic diagonalization equation (Eqn. 13) to relatively simple equations that can be applied at certain frequencies to obtain all the parameters relating the QPD readout to the optic's actual longitudinal displacement, pitch angular displacement and yaw angular displacement.