Diagonalization of Optical Lever (OpLev) signals for Type-B suspensions

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Created: 24 August 2018
Last update: 14 Oct 2018

Update Notes

14 Oct 2018

- For Type-B suspensions, the longitudinal displacement of the TM would cause change of beam spot in the vertical direction so I swapped (Long) and v in section 5 and 7. I also modified swapped some sin and cos so the mathematics is consistent throughout. Figures are also modified to match the change.
- Fixed some typos
- I worked on the Oplev of SR3 during the week of 8^{th} Oct, 2018. I applied the matrix (44) and (22) for tile and length sensing Oplev respectively without any rotational transformation, i.e. $\varphi_T = \varphi_L = \gamma = 0$, as they happened to be close to zero. But, because of the design of the suspension, pitch is known to be coupled with length. I didn't have enough time to study this effect so I couldn't verify my matrix but I expect it to be close to correct.

1 Introduction

This document was written after my lecture discussing the basics of optical lever (OpLev). The lecture slides were uploaded to JGWdoc and the document number is JGW-G1808874-v1. I changed some notations so equations seen in this document might not be consistent with that of the lecture. The reason for modifying the notations is to eliminate ambiguity in the slides and to make things more clear. It wasn't so clear because I was in a rush while I was preparing the lecture.

I wasn't able to work with the length sensing OpLev during my stay in KAGRA. So any discussion beyond the simple tilt sensing OpLev remains purely theoretical and untested. Therefore, I strongly encourage you to derive the equations by yourself instead of simply applying them. This is because I am, by no means, an expert in optics. So, my derivation might not be correct or I might misunderstand. More importantly, I need your contribution to verify the validity of this document. And if you find anything wrong in this document, I hope this would encourage you to correct my faults and propose another solution to this problem because this knowledge of diagonalizing OpLev signal is, for some reason, missing from the JGWdoc.

Just in case you accidentally opened this document and still wanted to finish reading anyways for whatever reason, OpLev is a external position sensing system that tells the longitudinal, yaw and pitch displacements of the test mass (TM). In the suspension system, we pair actuators with sensors so we can control the actuators. This is called closing the control loop. Due to varies reasons, we cannot use local sensors for the test mass so we need to use alternatives. Since the test mass is technically a mirror, we can direct a light beam to the TM and by interpreting the reflected beam we can tell its longitudinal, pitch and yaw displacements.

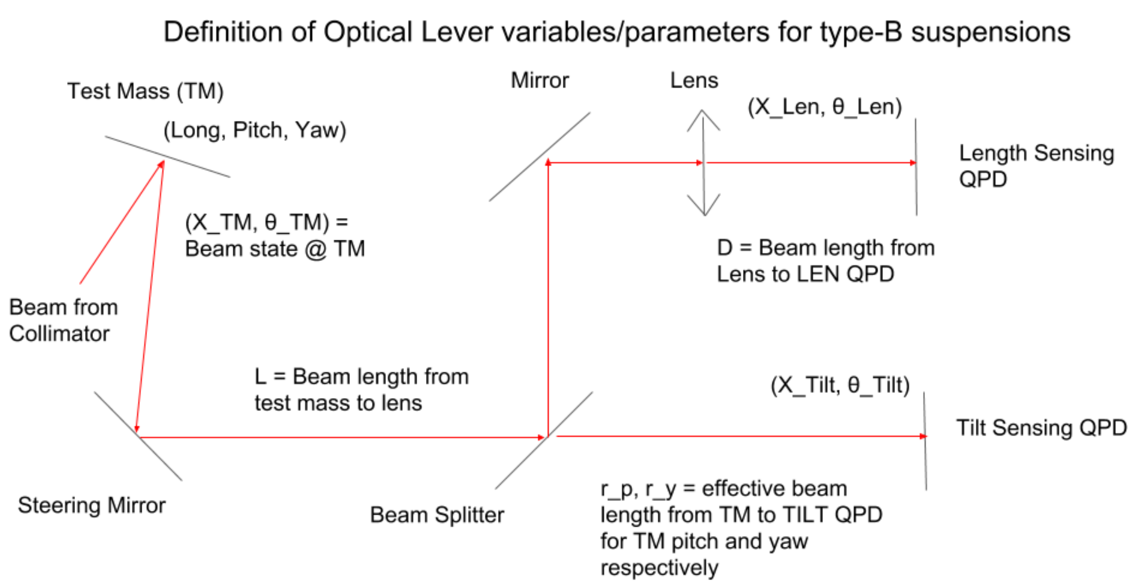
Please also don't forget to take a look on Simon's original document on OpLev setups and basic principles (JGW-T1605788-v9).

2 Prerequisites

If you find yourself not knowing at least one of the items from the list below, please try to learn so by reading my original lecture slides and Simon's document or learning from colleagues or from the Internet.

- Geometry and coordinate transformation (rotation in particular)
- Basic understanding in optical devices (lens, mirror and beam splitter)
- Matrix manipulation
- Quadrant photodiode (QPD) (calibration and defining its coordinate system)
- Defining coordinate system for the QPD in the MEDM screen
- Ray transfer matrix (RTM a.k.a ABCD matrix)
- Power spectrum and phase diagram from diaggui.

3 Definitions



From the picture, we can see the beam starts from the collimator. In reality, the beam will enter the suspension vacuum chamber from an angle such that the angle of incidence is non-zero. This non-zero angle of incidence is particularly important because, as we shall see later, with a zero angle we could never detect longitudinal displacement. The beam bounces off the test mass and meets the steering mirror which is used to direct the beam so that it goes through a beam splitter. One part of the beam will strike the tilt sensing QPD and the other part will meet a length sensing QPD after going through a lens with a focal length, f.

4 Tilt Sensing OpLev

by the definition of the ray transfer matrix, we can write:

$$\begin{pmatrix} X_{Tilt} \\ \theta_{Tilt} \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_{TM} \\ \theta_{TM} \end{pmatrix} \tag{1}$$

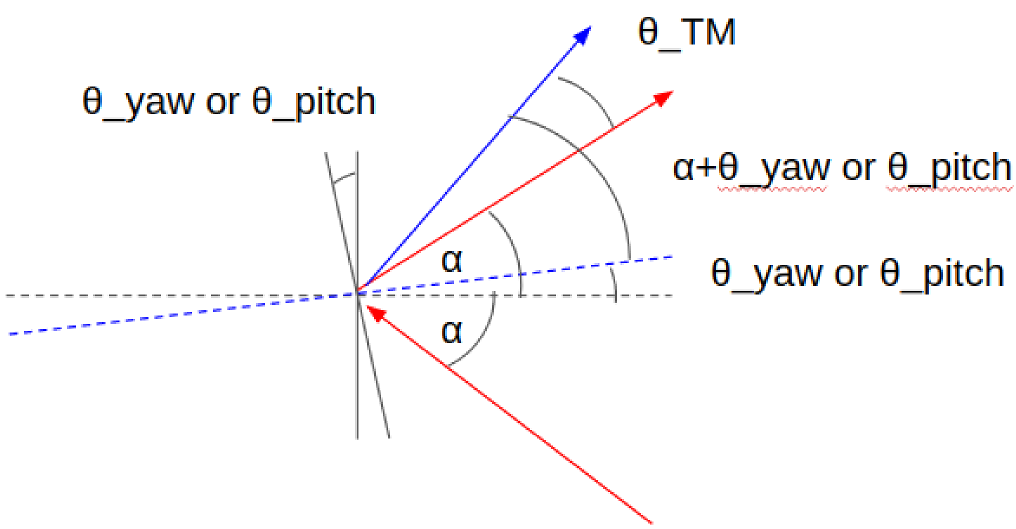
where

- X_{TM} and θ_{TM} are the displacement and the angle between the ray and optical axis at the TM plane,
- r is the beam length between the TM and the Tilt sensing QPD and
- X_{Tilt} and θ_{Tilt} are the displacement and the angle between the ray and optical axis at the Tilt QPD plane.

Since QPD only senses displacement, we can ignore the angle of the ray and write:

$$X_{Tilt} = X_{TM} + r\theta_{TM} \tag{2}$$

For simplicity, we will assume the contribution of X_{TM} is small compare to that of θ_{TM}.



The above picture shows the displacement of the OpLev beam corresponding to a tilt (yaw or pitch) of the TM. The red beam is the path of the original path and the blue beam is the reflected beam after the TM is tilted. Dotted lines are the corresponding normals of the TM. And, α is the angle of incidence. It is easy to see that:

for pure yaw:

$$\theta_{TM} = \alpha + \theta_{yaw} + \theta_{yaw} - \alpha \tag{3}$$

$$\theta_{TM} = 2\theta yaw \tag{4}$$

and for pure pitch:

$$\theta_{TM} = \alpha + \theta_{pitch} + \theta_{pitch} - \alpha \tag{5}$$

$$\theta_{TM} = 2\theta pitch \tag{6}$$

While we can safely assume pitch and yaw are axes independent of each other, and since the displacement seen at the tilt QPD plane is the effective length times the tilt angle, we can write rθ_{TM} in the vector form:

$$r\theta_{TM} = 2r_y\theta_{yaw} + 2r_p\theta_{pitch} \tag{7}$$

where r_y is the effective beam length of yaw from TM to tilt QPD and r_p is the effective beam length of pitch from TM to tilt QPD. And, $r_y\theta_{yaw}$ and $r_p\theta_{pitch}$ are two diagonal vectors at the tilt QPD plane.

then substituting (7) into (3), we get:

$$X_{Tilt} = \begin{pmatrix} 2r_y & 0 \\ 0 & 2r_p \end{pmatrix} \begin{pmatrix} \theta_{yaw} \\ \theta_{pitch} \end{pmatrix} \tag{8}$$

As for X_{Tilt}, it is just the coordinate at the tilt QPD plane given by the output of the QPD. So, we can redefine X_{Tilt}

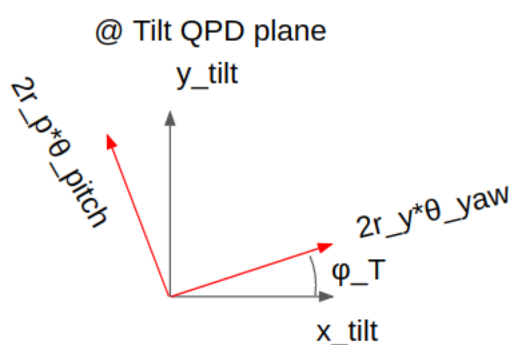
$$X_{Tilt} \equiv \begin{pmatrix} x_{tilt} \\ y_{tilt} \end{pmatrix} \tag{9}$$

where (x_{tilt},y_{tilt}) is the coordinate of the beam at the tilt QPD plane. And this coordinate can be calculated after the calibration of the tilt QPD, and it can be expressed by:

$$\begin{pmatrix} x_{tilt} \\ y_{tilt} \end{pmatrix} = \begin{pmatrix} C_{Tx} & 0 \\ 0 & C_{Ty} \end{pmatrix} \begin{pmatrix} N_{Tx} \\ N_{Ty} \end{pmatrix} \tag{10}$$

where C_{Tx} and C_{Ty} are the x and y axes calibration factors in [mm/counts] for the tilt QPD and N_{Tx} and N_{Ty} are the number of counts of QPD x and y direction output.

In principle, the x direction of the QPD measures roughly the yaw component and y direction measures roughly pitch. However, since the beam will bounce off a steering mirror before hitting the tilt QPD, a slight tilt in the steering mirror will rotate the image at QPD hence x and y do not necessarily align with yaw and pitch.



So, we need to introduce a rotational transformation from QPD coordinate system to yaw-pitch coordinate system.

and substituting (10) and (9) into (8), we get:

$$\begin{pmatrix} cos(\varphi_T) & sin(\varphi_T) \\ -sin(\varphi_T) & cos(\varphi_T) \end{pmatrix} \begin{pmatrix} C_{Tx} & 0 \\ 0 & C_{Ty} \end{pmatrix} \begin{pmatrix} N_{Tx} \\ N_{Ty} \end{pmatrix} = \begin{pmatrix} 2r_y & 0 \\ 0 & 2r_p \end{pmatrix} \begin{pmatrix} \theta_{yaw} \\ \theta_{pitch} \end{pmatrix} \tag{11}$$

and by multiplying the inverse of the beam length matrix and introducing a conversion factor, K, we get:

$$\begin{pmatrix} \theta_{yaw} \\ \theta_{pitch} \end{pmatrix} = K \begin{pmatrix} 1/2r_y & 0 \\ 0 & 1/2r_p \end{pmatrix} \begin{pmatrix} cos(\varphi_T) & sin(\varphi_T) \\ -sin(\varphi_T) & cos(\varphi_T) \end{pmatrix} \begin{pmatrix} C_{Tx} & 0 \\ 0 & C_{Ty} \end{pmatrix} \begin{pmatrix} N_{Tx} \\ N_{Ty} \end{pmatrix} \tag{12}$$

where K is 10^3 $[\mu m/mm]$ so the final unit of yaw and pitch is in micro-radians and φ_T is the angle between the 'yaw' axis and the x axis on the Tilt QPD plane. We will discuss later on how to obtain this φ in general.

Now we obtained a 2x2 matrix which convert number of counts into tilt. This matrix can be directly plugged into the OL2EUL matrix in the MEDM OpLev screen for a temporary tilt OpLev. This is only temporary because at the beginning of this section we eliminated X_{TM} in equation (2). So, in reality, each of the yaw and pitch we obtained from this OL2EUL matrix contained a little bit of X_{TM} which eventually we will need to get rid of. But, we will need to calculate this X_{TM} before we can subtract it from X_{Tilt}.

5 Length Sensing OpLev

Similar to that we have done for the tilt sensing OpLev, using the ray transfer matrix, we can write:

$$\begin{pmatrix} X_{Len} \\ \theta_{Len} \end{pmatrix} = \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_{TM} \\ \theta_{TM} \end{pmatrix} \tag{13}$$

where

- X_{Len} is the displacement of the ray at the Length QPD plane
- θ_{Len} is the angle between the ray and the optical axis at the Length QPD plane
- D is the distance between the lens and the Length QPD
- f is the focal length of the lens
- L is the distance between the TM and the lens
- X_{TM} is the displacement of the ray at the TM plane and
- θ_{TM} is the angle of the ray at the TM plane

simplifying:

$$\begin{pmatrix} X_{Len} \\ \theta_{Len} \end{pmatrix} = \begin{pmatrix} 1 - D/f & (1 - D/f)L + D \\ -1/f & 1 - L/f \end{pmatrix} \begin{pmatrix} X_{TM} \\ \theta_{TM} \end{pmatrix} \tag{14}$$

And again, since the QPD can only give information of the displacement of the beam, so we write the displacement:

$$X_{Len} = (1 - D/f)X_{TM} + ((1 - D/f)L + D)\theta_{TM} \tag{15}$$

Now, there are two particular position where we can place the QPD that would make either the displacement or the angle term vanishes.

if we place the QPD at $D = f$, the equation becomes:

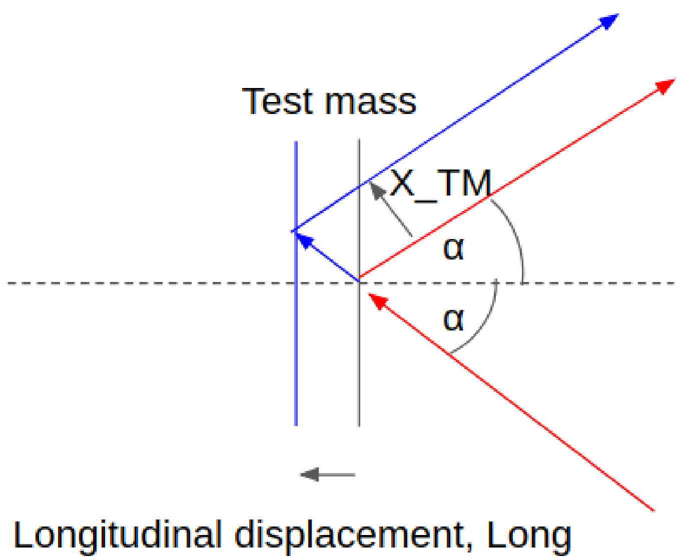
$$X_{Len} = f\theta_{TM} \tag{16}$$

and if we place the QPD at $D = Lf/(L-f)$, the equation becomes:

$$X_{Len} = (-f/(L-f))X_{TM} \tag{17}$$

So, we can place the QPD in these two positions so it is only sensitive to either tilt or longitudinal displacement. However, the argument for not using $D = f$ is that the sensitivity is low compared to the case we discussed as shown by equation (8). In practice, f is usually a few times smaller than r_y or r_p. So we tend to use the tilt sensing QPD without a lens because the sensitivity is higher that way.

However, for length sensing QPD, we can put it at $D = Lf/(L-f)$ so tilt information is filtered out and only longitudinal displacement information is left. And the relationship between longitudinal displacement and X_{TM} is discussed below:



From the above figure, X_{TM} is the displacement of the light beam when the test mass is shifted in the longitudinal direction. Blue ray is the ray after the shift while red ray is the original beam. From geometry, we can derive:

$$X_{TM} = 2(Long)sin(\alpha) \tag{18}$$

where $Long$ is the longitudinal displacement of the test mass and α is the angle of incidence.

As can be seen, if the angle of incidence becomes zero, X_{TM} will also become zero meaning that we can never detect see any longitudinal displacement and this is why it is mentioned that we must use a non-zero angle of incidence.

With that in mind, we substitute (18) into (17):

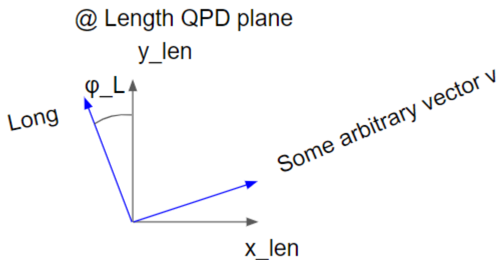
$$X_{Len} = (-f/(L-f))(2(Long)sin(\alpha)) \tag{19}$$

Similarly, X_{Len} is just the coordinate of the beam as seen from the output from the length QPD, so,

$$X_{Len} = \begin{pmatrix} x_{len} \\ y_{len} \end{pmatrix} = \begin{pmatrix} C_{Lx} & 0 \\ 0 & C_{Ly} \end{pmatrix} \begin{pmatrix} N_{Lx} \\ N_{Ly} \end{pmatrix} \tag{20}$$

Where (x_{len},y_{len}) is the coordinate [mm] of the beam spot given by the number of counts (N_{Lx},N_{Ly}) showed at the length QPD output which is calibrated with a calibration factor of (C_{L_x}, C_{Ly}) [mm/counts]

Again, the output of of the QPD due to longitudinal motion does not necessarily align with one of the axis of the QPD coordinate system. So, we need to use a rotational transformation to express the beam coordinate in the longitudinal axis while receiving QPD coordinates. However, in this case, since we are only interested in one vector, namely the longitudinal displacement, we will need to establish another arbitrary axis, v, for the rotational matrix to work. So, in principle, we will measuring something in the v channel but we are going to transform the coordinates so that signal in the v channel is minimized.



from the figure above, the *Long* axis is the "longitudinal" axis that we are interested in and we are getting outputs from the QPD in (x_{len}, y_{len}). So, we can apply a rotational transform:

$$\begin{pmatrix} cos(\varphi_T) & sin(\varphi_T) \\ -sin(\varphi_T) & cos(\varphi_T) \end{pmatrix} \begin{pmatrix} x_{len} \\ y_{len} \end{pmatrix} = X_{len} \tag{21}$$

substituting into (21) and (20) into (19),

$$\begin{pmatrix} v \\ Long \end{pmatrix} = K(f-L)/(2fsin(\alpha)) \begin{pmatrix} cos(\varphi_L) & sin(\varphi_L) \\ -sin(\varphi_L) & cos(\varphi_L) \end{pmatrix} \begin{pmatrix} C_{Lx} & 0 \\ 0 & C_{Ly} \end{pmatrix} \begin{pmatrix} N_{Lx} \\ N_{Ly} \end{pmatrix} \tag{22}$$

where, as a reminder, *Long* is the Longitudinal displacement, v is the arbitrary axis perpendicular to *Long*. K is the conversion factor [$\mu m/mm$]. f is the focal length of the lens, L is the distance between the Test mass and the lens, α is the incidence angle, φ_L is the angle between the *Long* axis and the QPD y axis, C_{Lx} and C_{Ly} are the calibration factors in length QPD x and y direction respectively (in [mm/counts]). N_{Lx} and N_{Ly} are the number of counts in the length QPD x and y axis respectively.

So now we obtained another equation relating longitudinal displacement and the number of counts read from the len QPD outputs. This transformation matrix can be plugged directly into the MEDM OL2EUL matrix to transform number of counts to displacement. Keep in mind the v is just an arbitrary axis so we cannot actually read anything from that channel with the MEDM screen. (We can only implement the first equation in (22) into the OL2EUL matrix, although it would be nice if we add another arbitrary channel v which to facilitate diagonalization.) Except we can firstly set φ_L to zero, so v aligns with x and we can use the x and y channel to represent the v and Long channel for further diagonalization purposes.

6 OpLev Diagonalization

So far I have only derived the equation needed for the OL2EUL matrix. There's an important parameter showed up in each equation, namely φ, which we still need to obtain. After obtaining that parameter, we will be able to derive the transformation matrixs applying on the QPDs xy outputs so that we can tell longitudinal, pitch and yaw separately. And, this process is called diagonalization. Without diagonalization, the raw signals, ($N_{Tx}, N_{Ty}, N_{Lx}, N_{Ly}$), which is weirdly named (TILT_YAW, TILT_PIT, LEN_YAW, LEN_PIT) in the MEDM OpLev screen, from QPDs each contain information about more than one variable, say in the case of tilt QPD, N_{Tx} will contain mostly yaw information but with a little bit of pitch information, so those channel will not be helpful for signal feedback. (Imagine controlling the yaw motion with signals containing pitch and yaw). So we need to decouple the degrees of freedom from each other and what we need to do is to look particular φ_T and φ_L in equation (22) and (12) so that the diagonalized channels $(Long, \theta_{pitch}, \theta_{yaw})$ tell exactly long, pitch and yaw. In short, we get coordinates in x and y and we apply transformation to that coordinate to obtain long (in the case of length QPD) or pitch and yaw separately (in the case of tilt QPD)

Mathematically, we need a 3x4 transformation matrix to transform the raw signals:

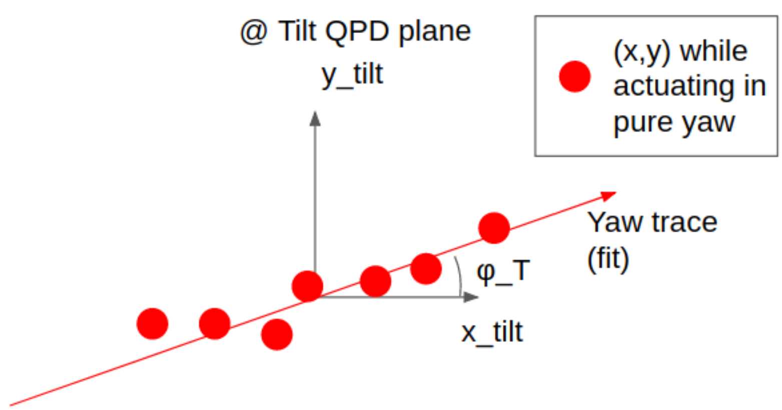
$$\begin{pmatrix} Long \\ \theta_{Pitch} \\ \theta_{yaw} \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \end{pmatrix} \begin{pmatrix} N_{Tx} \\ N_{Ty} \\ N_{Lx} \\ N_{Ly} \end{pmatrix} \tag{23}$$

We derived the equations for each A, B, C...L which are inside the equations (22) and (12) (Although (12) is not technically correct because it doesn't decouple pitch and yaw from Longitudinal displacement, we will assume it is correct and keep using the purpose of this section)

In principle, we can diagonalize the signals in two ways, by setting DC offsets or by measuring power spectrum.

For the former method, we simply inject DC actuations to the degree of freedom that we are interested in so that it moves purely in that freedom and we measure the changes with the non-diagonalized OpLev. By actuating the degrees of freedom separate, we can plot the separate beam traces in the QPDs coordinate systems and, in theory, the particular trace will be aligned with the axis corresponding to the degrees of freedom we were interested in. And then, we can simply calculate the angle between that trace and x or y axis and that angle shall be φ_T or φ_L.

To exemplify, assume we actuate the test mass in pure yaw direction. Doing so, by monitoring the tilt QPD signals, we can plot:



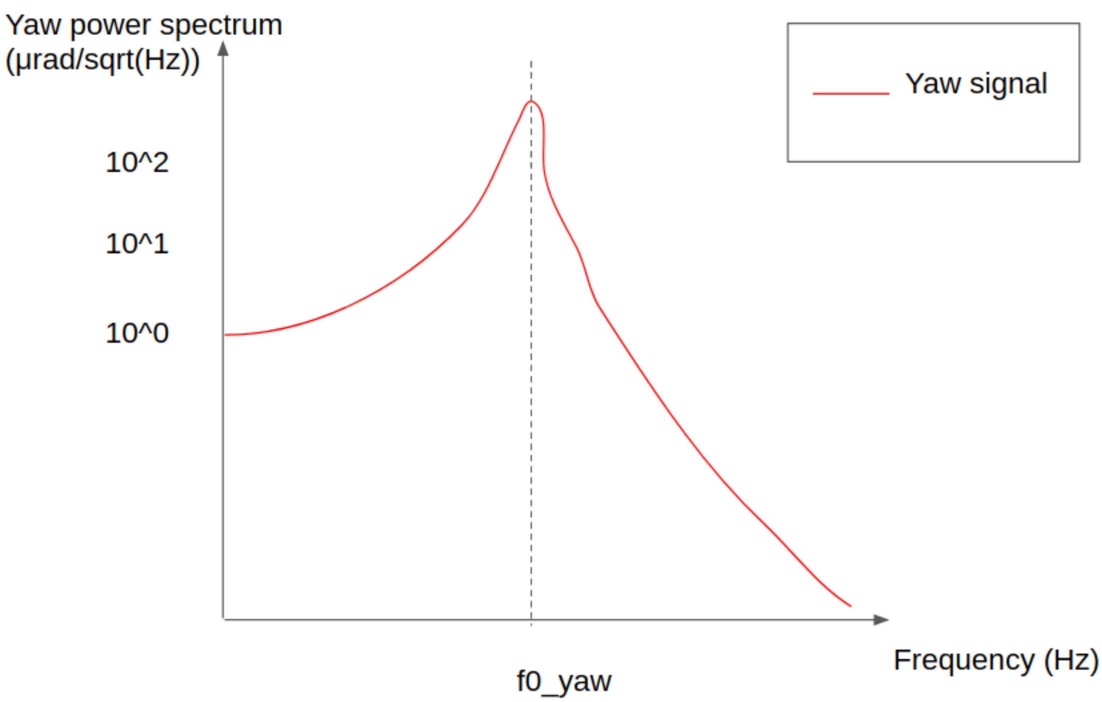
Where the red dots are the data points taken the tilt QPD channel while actuating yaw and the red trace is the linear fit of those data points. In principle, this trace will be aligned with the actual "yaw" axis in the tilt QPD frame. So, the angle is just the arctan of the slope of this line.

As can be seen, this method is very easy to implement and what we have to do is simply actuating the TM in yaw and in longitudinal direction in order to obtain φ_T and φ_L respectively.

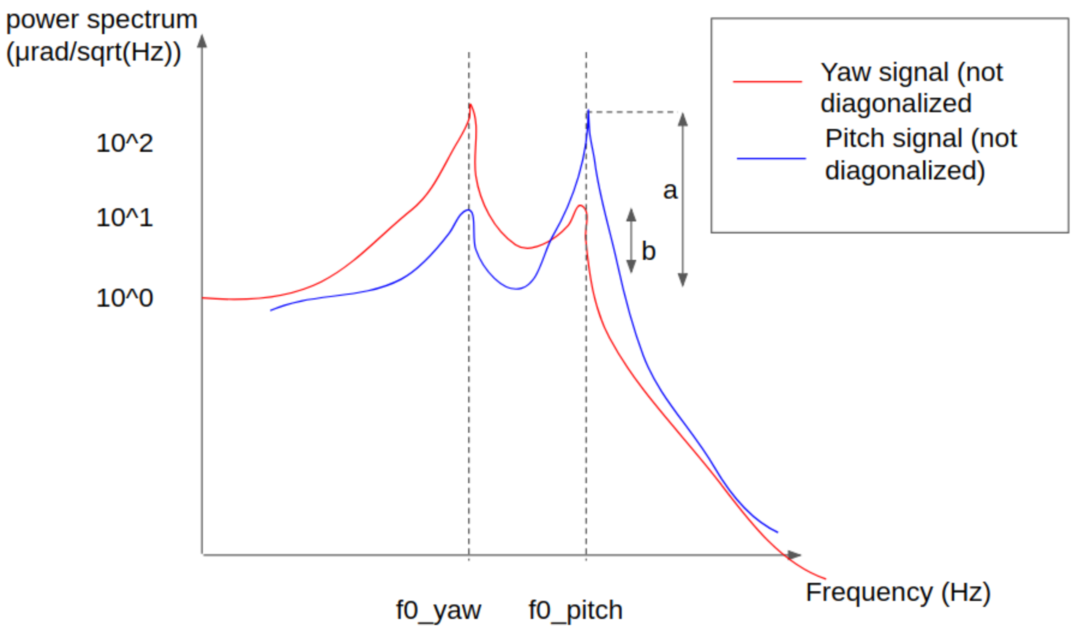
Yet, this accuracy of this method will not be great. This is because pure direction actuation is not always guaranteed. Although we have all actuators calibrated, when we put them into the actual suspension system, we cannot make sure that there is no error and that they actuate the system as they should, given a certain input signal. If we assume they can drive the system accurately according to a particular input signal, we wouldn't need sensors for them at all. So, we need sensors for actuators so that they can actuate the TM in pure yaw. But we didn't have the sensors, namely the OpLev we are now trying to setup. Therefore, we this pure yaw actuation is not very accurate and in fact it might have driven the TM in all directions besides yaw. Although it is said that this method of diagonalization is not very precise, this method should be included into the standard procedure so to estimate roughly the angle between the axes which can be compared with the results we will obtain later with the power spectrum method.

The power spectrum method is more accurate in determining the rotational transformation for the diagonalization matrix simply because of the fact that the test mass vibrates in different of degrees of freedom with distinguishable natural frequencies. With the help of the simulation software SUMCOM, we are able to pinpoint the eigenmodes and the eigen-frequencies that we are interested in (in this case, TM longitudinal, pitch and yaw). In theory, we should see a peak at a particular frequency if we plot power spectrum the signal detecting that degree of freedom while exciting it along with all other degrees of freedom that it might be coupled to.

Ideally, if the signal is perfectly diagonalized, the yaw channel will only measure yaw and the plot should look like the following red line:



where f_{0yaw} is the corresponding eigen-frequency of the yaw eigenmode and f_{0pitch} is the eigen-frequency of the pitch eigenmode.



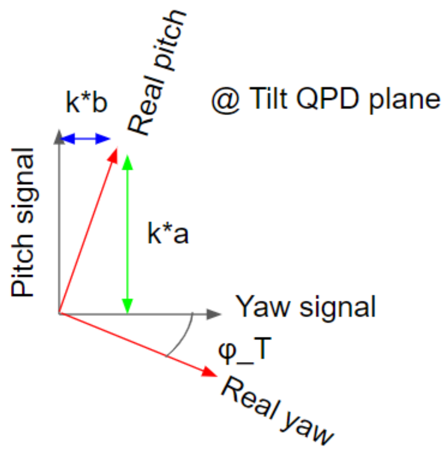
However, a non-diagonalized signal will give you two peaks, one at its own resonance frequency and one at the frequency of the degree of freedom it is coupled with.

Let's go back to the previous section where I derived the diagonalization matrix (equation (12)) with a rotational transformation. Let's also say that we set φ_T to zero and then began measuring the power spectrum of TM yaw and pitch while introducing noise to the system and it ended up showing the result exactly like the above blue and red power spectrum.

Blue line is the power spectrum of the pitch signal and red line is the power spectrum of the yaw signal. at frequency equals to f_{0pitch}, there is a small peak in the red line while a larger peak for the pitch signal. This means that when test mass vibrates in the pitch mode, there is some "a" amount of magnitude of pitch that is detected in the pitch channel while a "b" magnitude of pitch that is shown up the yaw channel. In an ideal case, all pitch motion should show up in the pitch channel. However, some of them ended up in the yaw signal.

Also please note that the ground reference for measuring "a" and "b" is almost arbitrary. In theory, we want to measure the height of the peaks from it's original height without the peak so we are measuring the additional effect of coupling. But since we cannot say we are sure what is the original height, so I am just going to say the reference of "a" and "b" is just roughly at the original place without the height. For "a", it doesn't matter very much since the power spectrum is plotted in log scale. In effect, one end is at some 10^2 magnitude and the other end is in the order of 10^0. So, you can measure it from 10^0 or $10^{0.1}$ and the difference is minimal. But for "b", that is the coupling from pitch to yaw, since both ends are in the order of 10^0 or 10^1, we need to guess the original magnitude of that power spectrum without any coupling. But, from experience, using the cursor function in diaggui and estimating the orignal position using interpolation roughly is probably fine to decouple the degrees of freedom from obvious to unnoticeable.

From the result of the above power spectrum, we can tell that the the real pitch and yaw axis are tilted relative to signal axis like:



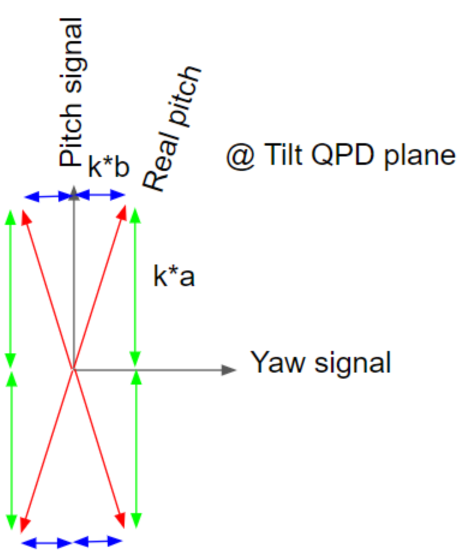
Where k is the 2 times effect beam length of the pitch (don't get confused by this k, if you don't understand why it exists at this point then just ignore it). a and b are just the height of the peaks of pitch and yaw signal respectively at f_{0pitch}. The red axes are the axes where the real pitch and yaw motion would cause the beam to trace at the tilt QPD plane. Pitch signal and Yaw signal are just "θ_{pitch}" and "θ_{yaw}" we derived in (12). So, from this figure, we can derive the rotational angle that we need

$$\varphi_T = -\arctan(kb/ka) = -\arctan(b/a) \tag{24}$$

It is negative because φ_T is positive counter-clockwise as defined. Note that the k constant doesn't matter because it will just cancel out itself in the end. But this is not the end of the story because what we can tell from the power spectrum is that when the test mass moves in pitch direction, "b" amount will show up in the yaw signal and "a" amount will show up in the pitch signal but it doesn't tell really tell the direction of "a" and "b". So, mathematical we have four solutions of φ_T.

$$\varphi_T = \pm\arctan(b/a) or \pm\arctan(b/a) + \pi \tag{25}$$

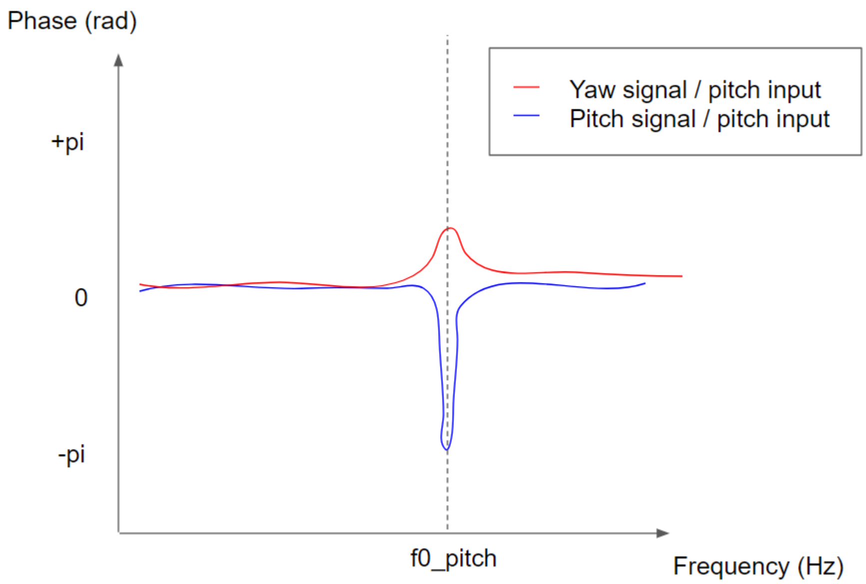
As in,



All four rotations will give "a" magnitude of pitch showing in pitch channel and "b" magnitude of pitch showing up in yaw channel. To obtain the real case, first we can eliminate by defining the direction of x_{tilt} and y_{tilt} axis of the QPD so we can always roughly align positive pitch and positive yaw with positive x_{tilt} and y_{tilt} (or y_{tilt} and x_{tilt} if it is more convenient). So, the latter two solutions can be always be rejected by aligning the axes manually.

For the former solutions, $\pm\arctan(b/a)$, we can note that for $\varphi_T = +\arctan(b/a)$, real pitch will give positive pitch signal and negative yaw signal. While for the case of $\varphi_T = -\arctan(b/a)$ real pitch will give positive pitch signal and positive yaw signal. So, we can distinguish by looking at the phase diagram of the transfer function (assuming we are exciting the pitch and yaw motion with pitch and yaw input).

For example if we obtain this kind of result:



That means whatever real pitch the yaw channel is measuring, is opposite to that of the pitch channel. That means we can decide $\varphi_T = +\arctan(b/a)$ because that is the case where positive real pitch will show up as positive pitch and negative yaw in our signals. Conversely, if the red trace in the phase diagram follows the blue line towards $-\pi$, that means the other way around and $\varphi_T = -\arctan(b/a)$, which would mean positive real pitch shows up as positive pitch and yaw in the digital system.

So after applying the rotational transformation using equation (12) (and (22) for the longitudinal case), we should repeat the process to make sure we are measuring real pitch and real yaw only in their corresponding signal channels.

7 Longitudinal to Tilt Coupling

With the tools in sections above, we can make sure that the longitudinal channel measures maximal longitudinal, pitch channel doesn't measure yaw and yaw channel doesn't measure pitch. However, in the beginning I cheated and say

$$X_{Tilt} = r\theta_{TM} \tag{26}$$

recall X_{Tilt} is the displacement of the OpLev beam showing up at the tilt QPD plane and $\theta_T M$ is the angle between the original OpLev beam at the TM plane and the OpLev beam when the test mass is tilted. And r is the effective beam length from the TM to tilt QPD. However, to be completely accurate, from the definition of ray transfer matrix

$$\begin{pmatrix} X_{Tilt} \\ \theta_{Tilt} \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_{TM} \\ \theta_{TM} \end{pmatrix} \tag{27}$$

so,

$$X_{Tilt} = X_{TM} + r\theta_{TM} \tag{28}$$

where X_{TM} is really the parallel displacement of the OpLev beam at the TM plane. And, X_{TM} is only related to longitudinal displacement, $X_{TM} = 2(Long)\sin(\alpha)$, so, recalling from (22):

$$\big(Long\big) = (f - L)/(2fsin(\alpha)) \begin{pmatrix} -sin(\varphi_L) & cos(\varphi_L) \end{pmatrix} \begin{pmatrix} C_{Lx} & 0 \\ 0 & C_{Ly} \end{pmatrix} \begin{pmatrix} N_{Lx} \\ N_{Ly} \end{pmatrix} \tag{29}$$

ignore the conversion factor K for the moment substituting (29) into $X_{TM} = 2(Long)\sin(\alpha)$,

$$\big(X_{TM}\big) = ((f - L)/f) \begin{pmatrix} -sin(\varphi_L) & cos(\varphi_L) \end{pmatrix} \begin{pmatrix} C_{Lx} & 0 \\ 0 & C_{Ly} \end{pmatrix} \begin{pmatrix} N_{Lx} \\ N_{Ly} \end{pmatrix} \tag{30}$$

With that, (7) and (10) we can rewrite (28) into

$$\begin{pmatrix} C_{Tx} & 0 \\ 0 & C_{Ty} \end{pmatrix} \begin{pmatrix} N_{Tx} \\ N_{Ty} \end{pmatrix} = ((f-L)/f) \begin{pmatrix} -sin(\varphi_L) & cos(\varphi_L) \end{pmatrix} \begin{pmatrix} C_{Lx} & 0 \\ 0 & C_{Ly} \end{pmatrix} \begin{pmatrix} N_{Lx} \\ N_{Ly} \end{pmatrix} + \begin{pmatrix} 2r_y & 0 \\ 0 & 2r_p \end{pmatrix} \begin{pmatrix} \theta_{yaw} \\ \theta_{pitch} \end{pmatrix} \tag{31}$$

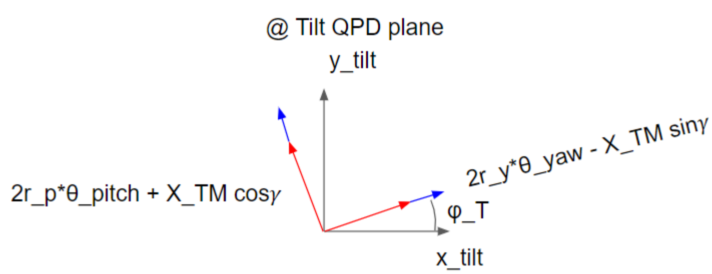
Simplifying,

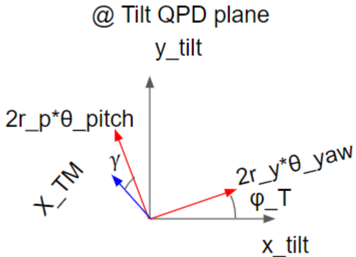
$$\begin{pmatrix} C_{Tx} & 0 \\ 0 & C_{Ty} \end{pmatrix} \begin{pmatrix} N_{Tx} \\ N_{Ty} \end{pmatrix} = ((f - L)/f) \begin{pmatrix} -C_{Lx}sin(\varphi_L) & C_{Ly}cos(\varphi_L) \end{pmatrix} \begin{pmatrix} N_{Lx} \\ N_{Ly} \end{pmatrix} + \begin{pmatrix} 2r_y & 0 \\ 0 & 2r_p \end{pmatrix} \begin{pmatrix} \theta_{yaw} \\ \theta_{pitch} \end{pmatrix} \tag{32}$$

Apply rotational transformation to X_{Tilt} so we can express X_{Tilt} in yaw-pitch coordinate frame, with more simplification:

$$\begin{pmatrix} C_{Tx}cos(\varphi_T) & C_{Ty}sin(\varphi_T) \\ -C_{Tx}sin(\varphi_T) & C_{Ty}cos(\varphi_T) \end{pmatrix} \begin{pmatrix} N_{Tx} \\ N_{Ty} \end{pmatrix} = ((f-L)/f) \begin{pmatrix} -C_{Lx}sin(\varphi_L) & C_{Ly}cos(\varphi_L) \end{pmatrix} \begin{pmatrix} N_{Lx} \\ N_{Ly} \end{pmatrix} + \begin{pmatrix} 2r_y & 0 \\ 0 & 2r_p \end{pmatrix} \begin{pmatrix} \theta_{yaw} \\ \theta_{pitch} \end{pmatrix} \tag{33}$$

Comparing this equation to (12), this is effectively saying that for the previous measurements, there are some components of X_{TM} that was added into the yaw and pitch that we were measuring:





So, what we should really do is break X_{TM} into components form so it is expressed in the same basis of yaw and pitch. And then, subtract it from the measurments X_{Tilt}. For convenience, we will simplify X_{TM} and then add also N_{Tx} and N_{Ty} so X_{Tilt} can subtract it.

$$X_{TM} = ((f-L)/f)\begin{pmatrix} C_{Lx}cos(\varphi_L) & C_{Ly}sin(\varphi_L)\end{pmatrix}\begin{pmatrix} N_{Lx} \\ N_{Ly}\end{pmatrix} \tag{34}$$

$$= ((f-L)/f)\begin{pmatrix} -sin(\gamma) \\ cos(\gamma)\end{pmatrix}\begin{pmatrix} -C_{Lx}sin(\varphi_L) & C_{Ly}cos(\varphi_L)\end{pmatrix}\begin{pmatrix} N_{Lx} \\ N_{Ly}\end{pmatrix} \tag{35}$$

$$= ((f-L)/f)\begin{pmatrix} C_{Lx}sin(\varphi_L)sin(\gamma) & -C_{Ly}cos(\varphi_L)sin(\gamma) \\ -C_{Lx}sin(\varphi_L)cos(\gamma) & C_{Ly}cos(\varphi_L)cos(\gamma)\end{pmatrix}\begin{pmatrix} N_{Lx} \\ N_{Ly}\end{pmatrix} \tag{36}$$

$$= ((f-L)/f)\begin{pmatrix} 0 & 0 & C_{Lx}sin(\varphi_L)sin(\gamma) & -C_{Ly}cos(\varphi_L)sin(\gamma) \\ 0 & 0 & -C_{Lx}sin(\varphi_L)cos(\gamma) & C_{Ly}cos(\varphi_L)cos(\gamma)\end{pmatrix}\begin{pmatrix} N_{Tx} \\ N_{Ty} \\ N_{Lx} \\ N_{Ly}\end{pmatrix} \tag{37}$$

and for X_{Tilt}

$$X_{Tilt} = \begin{pmatrix} C_{Tx}cos(\varphi_T) & C_{Ty}sin(\varphi_T) \\ -C_{Tx}sin(\varphi_T) & C_{Ty}cos(\varphi_T)\end{pmatrix}\begin{pmatrix} N_{Tx} \\ N_{Ty}\end{pmatrix} \tag{38}$$

$$= \begin{pmatrix} C_{Tx}cos(\varphi_T) & C_{Ty}sin(\varphi_T) & 0 & 0 \\ -C_{Tx}sin(\varphi_T) & C_{Ty}cos(\varphi_T) & 0 & 0\end{pmatrix}\begin{pmatrix} N_{Tx} \\ N_{Ty} \\ N_{Lx} \\ N_{Ly}\end{pmatrix} \tag{39}$$

So, the final assembly would be:

$$r\theta_{TM} = X_{Tilt} - X_{TM} \tag{40}$$

$$= \begin{pmatrix} C_{Tx}cos(\varphi_T) & C_{Ty}sin(\varphi_T) & -((f-L)/f)C_{Lx}sin(\varphi_L)sin(\gamma) & ((f-L)/f)C_{Ly}cos(\varphi_L)sin(\gamma) \\ -C_{Tx}sin(\varphi_T) & C_{Ty}cos(\varphi_T) & ((f-L)/f)C_{Lx}sin(\varphi_L)cos(\gamma) & -((f-L)/f)C_{Ly}cos(\varphi_L)cos(\gamma)\end{pmatrix}\begin{pmatrix} N_{Tx} \\ N_{Ty} \\ N_{Lx} \\ N_{Ly}\end{pmatrix} \tag{41}$$

$$\equiv \begin{pmatrix} A & B & C & D \\ E & F & G & H\end{pmatrix}\begin{pmatrix} N_{Tx} \\ N_{Ty} \\ N_{Lx} \\ N_{Ly}\end{pmatrix} \tag{42}$$

so applying the inverse of r and introducing the conversion factor, we get:

$$\begin{pmatrix}\theta_{yaw}\\ \theta_{pitch}\end{pmatrix} = K\begin{pmatrix}1/(2r_y) & 0\\ 0 & 1/(2r_p)\end{pmatrix}\begin{pmatrix}A & B & C & D\\ E & F & G & H\end{pmatrix}\begin{pmatrix}N_{Tx}\\ N_{Ty}\\ N_{Lx}\\ N_{Ly}\end{pmatrix} \tag{43}$$

$$\begin{pmatrix}\theta_{yaw}\\ \theta_{pitch}\end{pmatrix} = K\begin{pmatrix}1/(2r_y) & 0\\ 0 & 1/(2r_p)\end{pmatrix}\begin{pmatrix}C_{Tx}cos(\varphi_T) & -C_{Tx}sin(\varphi_T)\\ C_{Ty}sin(\varphi_T) & C_{Ty}cos(\varphi_T)\\ -((f-L)/f)C_{Lx}sin(\varphi_L)sin(\gamma) & ((f-L)/f)C_{Lx}sin(\varphi_L)cos(\gamma)\\ ((f-L)/f)C_{Ly}cos(\varphi_L)sin(\gamma) & -((f-L)/f)C_{Ly}cos(\varphi_L)cos(\gamma)\end{pmatrix}^T\begin{pmatrix}N_{Tx}\\ N_{Ty}\\ N_{Lx}\\ N_{Ly}\end{pmatrix} \tag{44}$$

I take the transpose of that matrix so I can show the actual terms in the equation.

Note that γ is the angle between the X_{TM} axis and the "pitch" axis. This angle can be found using the method described in the diagonalization section. What we have to do is to plot the power spectrum of TM pitch and yaw channel while at least exciting TM in longitudinal direction. And you know what to do next.

In summary, use (12) for Tilt sensing OpLev before we get a Length sensing OpLev and use (22) for the Length sensing OpLev and finally use (44) Tilt sensing OpLev.

And this shall be the end of story, except when we consider the misplacement of the Length sensin OpLev, there will be some Tilt to longitudinal coupling and we might want to decouple them. But I will leave the derivation to you since 1) the coupling effect shall be minimal, 2) if you understand the derivation in this document, you will have no problem deriving the final ultimate equation for the OL2EUL matrix.

8 Minor Notes

Reminder: N_{Tx}, N_{Ty}, N_{Lx} and N_{Ly} really refers to TILT_YAW, TILT_PIT, LEN_YAW and LEN_PIT in the MEDM screen respectively. Or, not respectively if you like to define weird coordinate system for the QPD.

My email: astrotec@connect.hku.hk or 1155116690@link.cuhk.edu.hk Please notify me if you find anything wrong, or if you find my equation works.