Bayesian approach to the locking problem for high finesse suspended optical cavities

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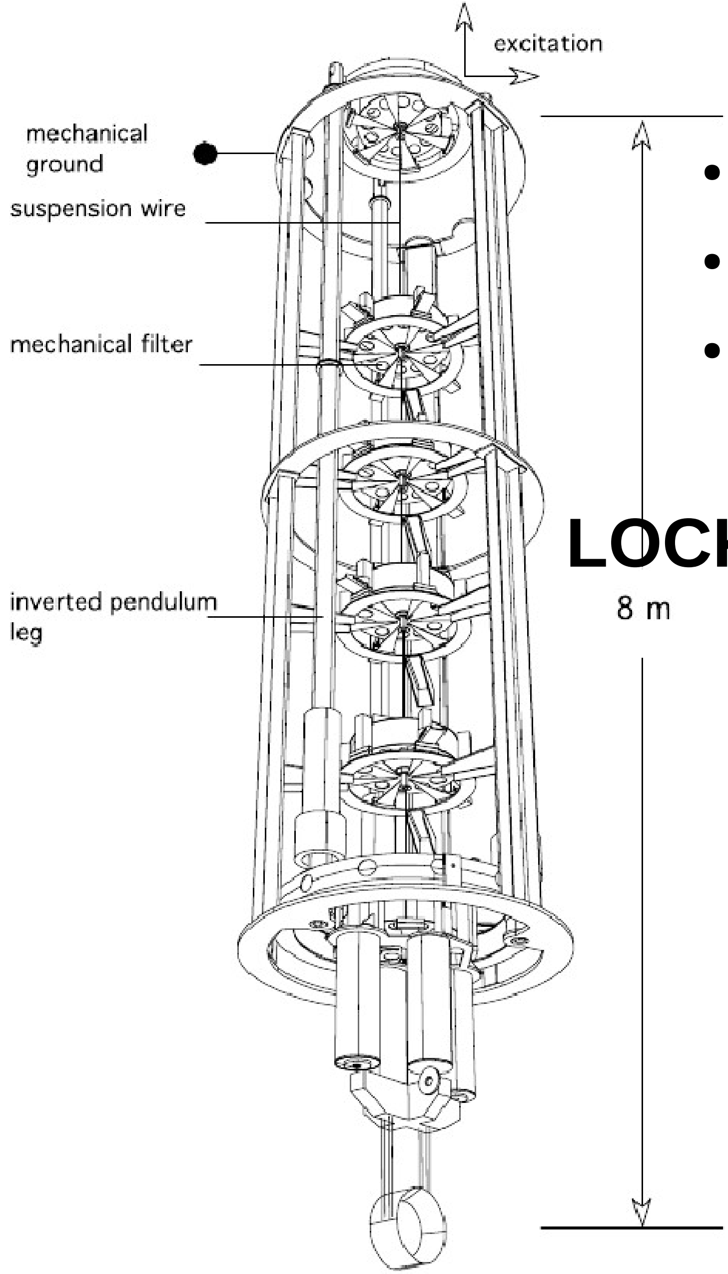
Introduction



- Suspensions are not efficient below a few Hz
- If no force applied → tipical rms 10^{-6} m
- Time the cavity spends in resonance $\tau \simeq \frac{\lambda_\ell}{v\mathcal{F}}$

LOCKING ACQUISITION – Classical approach

Cavity state is known only at resonance

↓

Force is applied only at resonance

PROBLEM: $$F_{max} \simeq \frac{mv}{\tau} = \frac{mv^2}{\lambda_\ell}\mathcal{F}$$

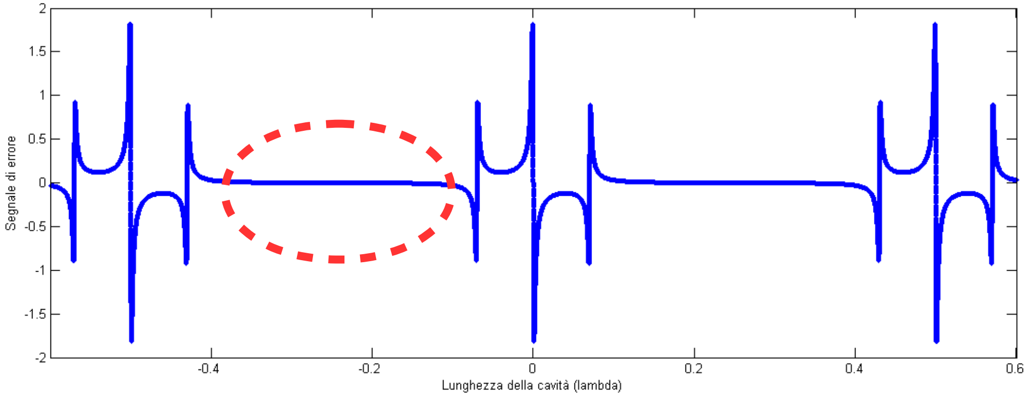
THE IDEA:

Still apply a force out of resonance

- More time available
  → Less force to apply
  → Less noise injected
- Faster locking acquisition

THE POINT:

Are we completely ignorant out of resonance?



- We know the dynamics
- We can make an estimation of the state
- We can continue applying a force

Bayesian approach

MEASUREMENT $P\left(\boldsymbol{y} \mid \boldsymbol{x}\right)$

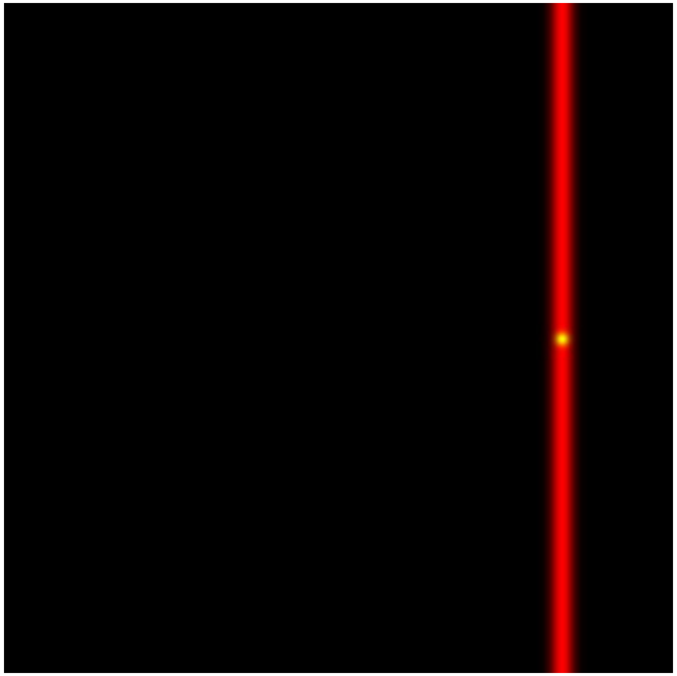
Parameters State

A PRIORI $P\left(\boldsymbol{x}, t \mid \boldsymbol{x}', t'\right)$

(Dynamics)

PREDICTION STEP

$$P\left(\boldsymbol{x}, t \mid \boldsymbol{y}_1, t_1; \cdots; \boldsymbol{y}_n, t_n\right) = \int P\left(\boldsymbol{x}, t \mid \boldsymbol{x}_n, t_n\right) P\left(\boldsymbol{x}_n, t_n \mid \boldsymbol{y}_1, t_1; \cdots; \boldsymbol{y}_n, t_n\right) \mathrm{d}\boldsymbol{x}_n$$



State space
V vs X

UPDATE STEP

$$P\left(\boldsymbol{x}_{n+1}, t_{n+1} \mid \boldsymbol{y}_1, t_1; \cdots; \boldsymbol{y}_{n+1}, t_{n+1}\right) = \frac{P\left(\boldsymbol{y}_{n+1} \mid \boldsymbol{x}_{n+1}\right) P\left(\boldsymbol{x}_{n+1}, t_{n+1} \mid \boldsymbol{y}_1, t_1; \cdots; \boldsymbol{y}_n, t_n\right)}{\int P\left(\boldsymbol{y}_{n+1} \mid \boldsymbol{x}_{n+1}\right) P\left(\boldsymbol{x}_{n+1}, t_{n+1} \mid \boldsymbol{y}_1, t_1; \cdots; \boldsymbol{y}_n, t_n\right) \mathrm{d}\boldsymbol{x}_{n+1}}$$

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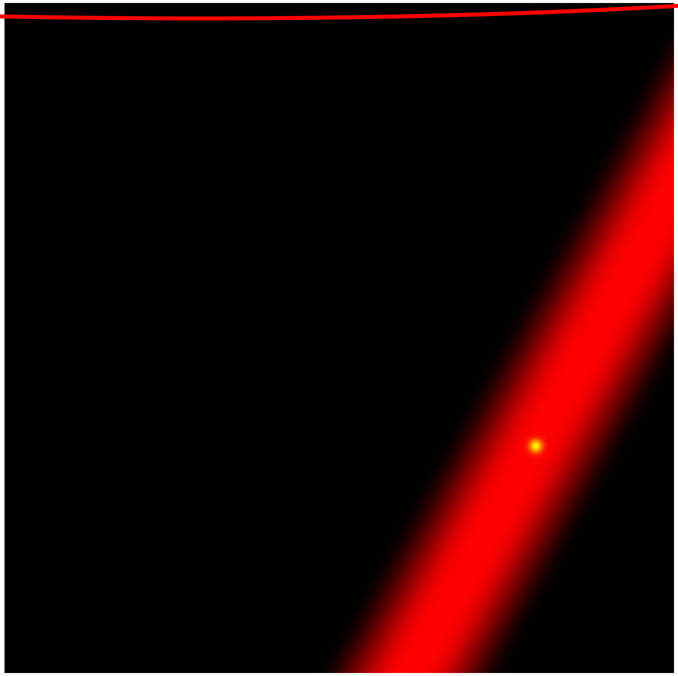

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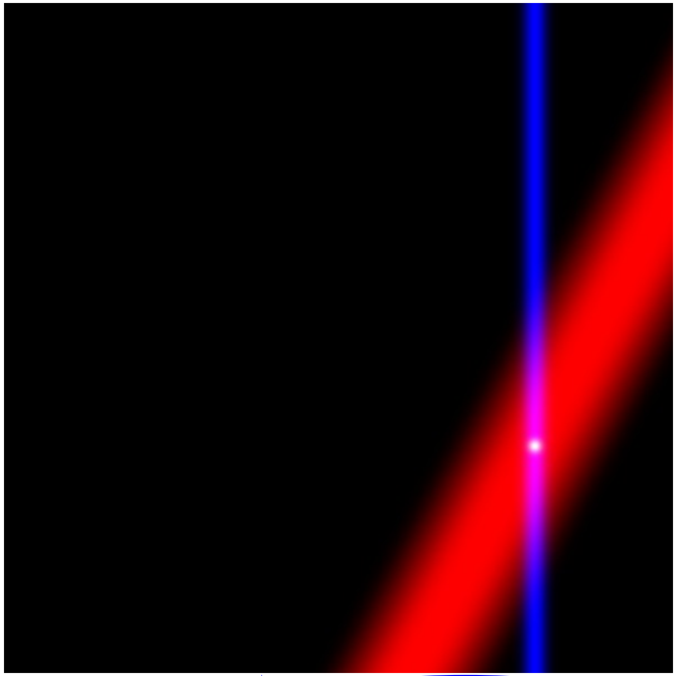

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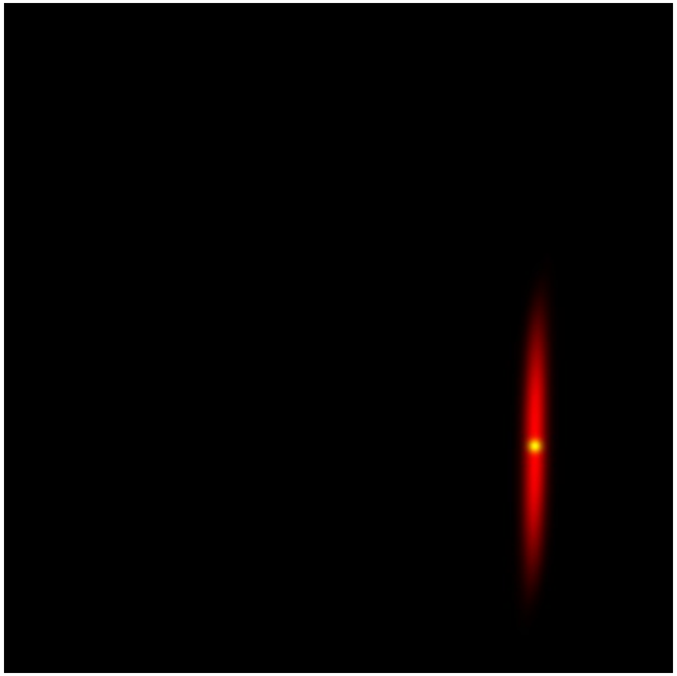

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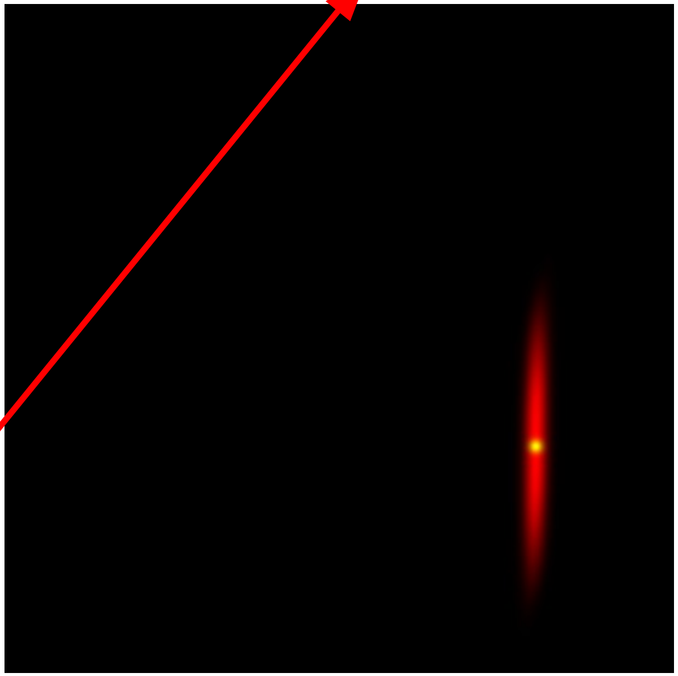

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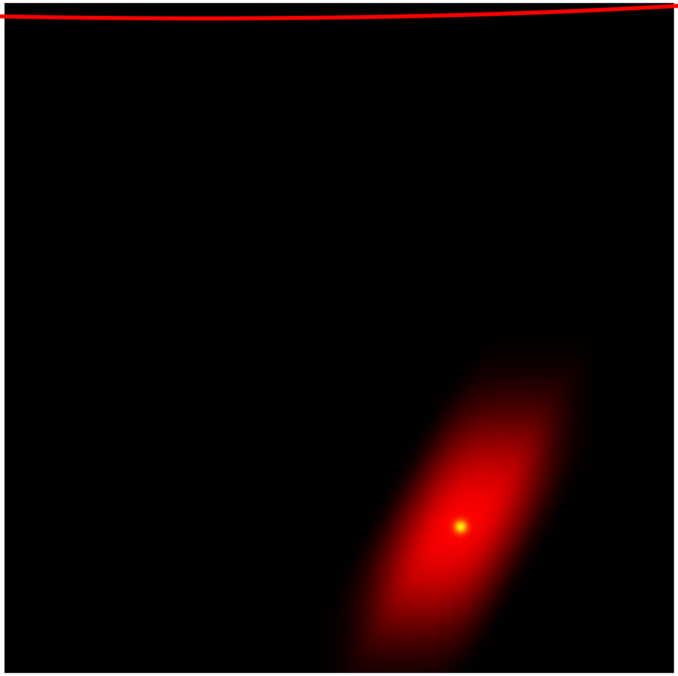

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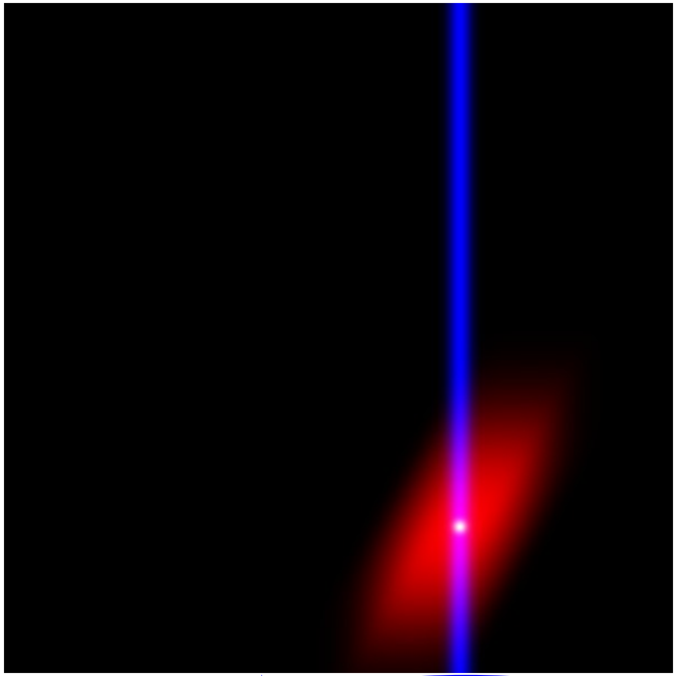

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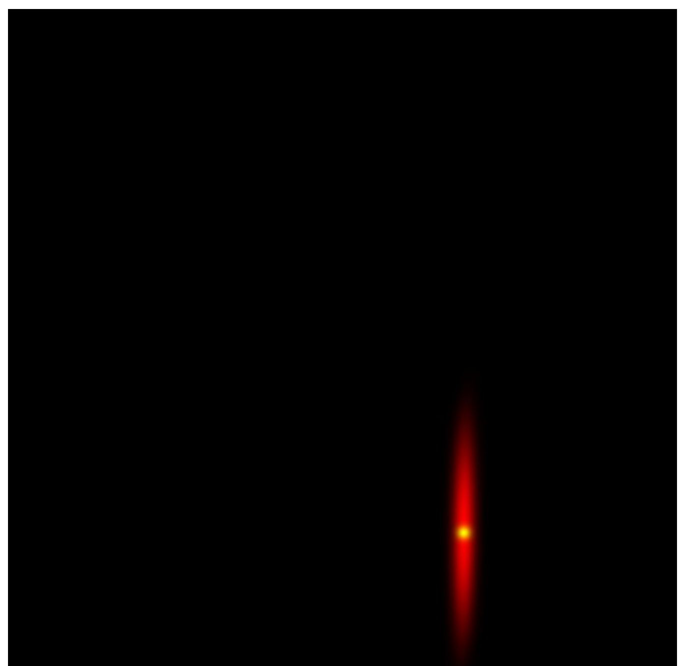

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Simplyfied model

Position and velocity

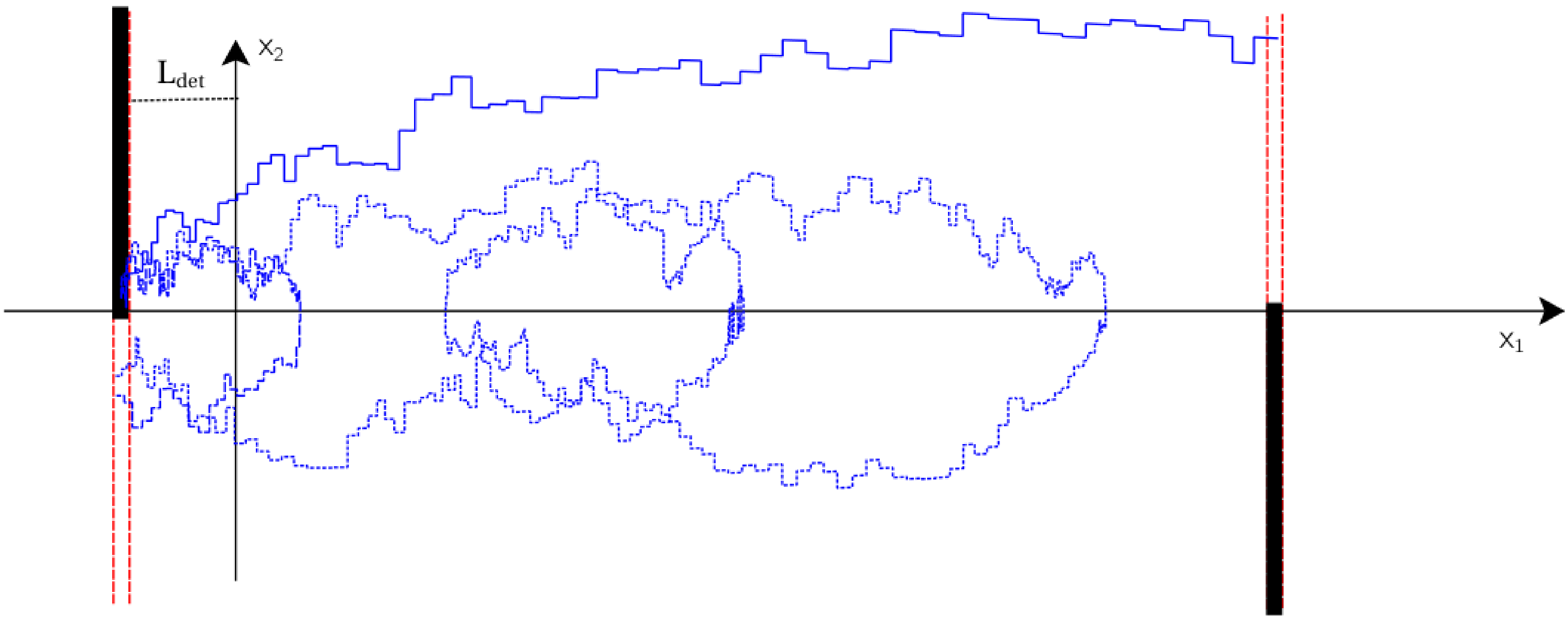
Wiener process

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x})\,\mathrm{d}t + \boldsymbol{\beta}\,\mathrm{d}W$$

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\gamma \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

Harmonic oscillator

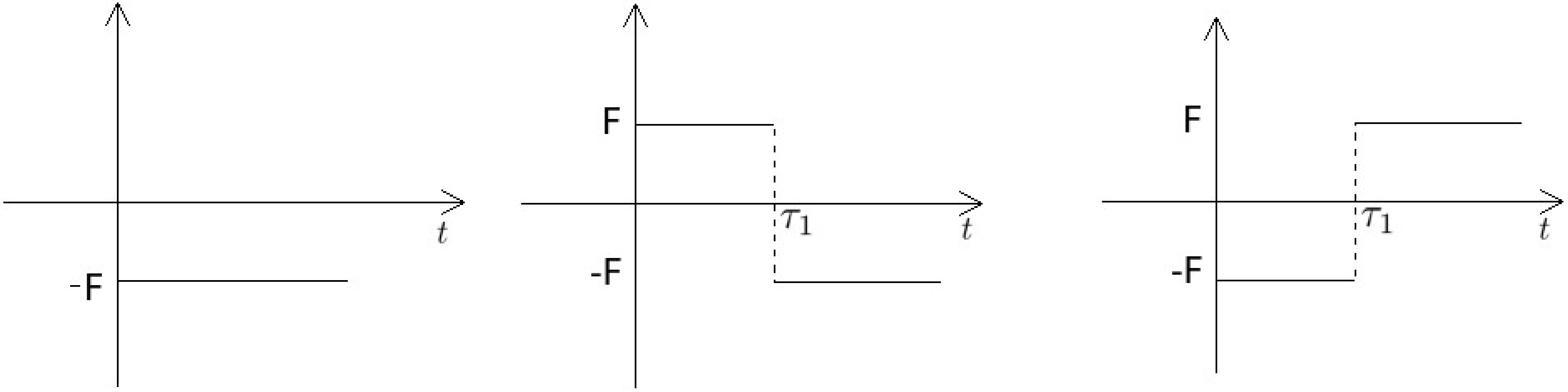
External force



- Which external force do we apply to slow down the cavity?

Locking strategies

- Setting a maximal force to push mirrors
- Optimal feedback without noise and without input filtering



- We filter the input force with a low pass filter

$$f(t) = F_0 \left[g(t) - 2g\left(t - \tau_1\right)\right]$$

Unit step response

Parameter

- And see what happens

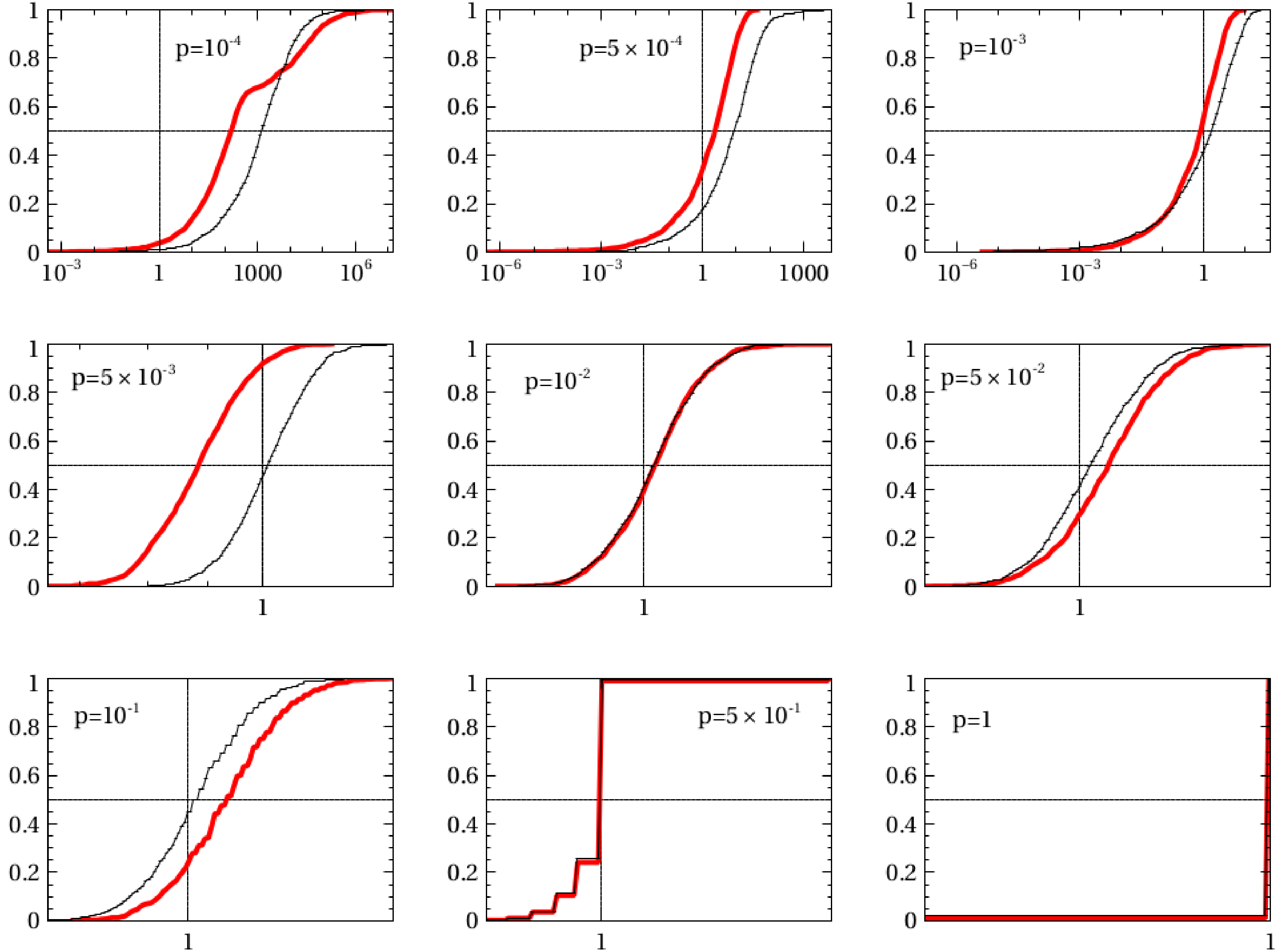
Numerical simulation

- Evolution of many cavity states from the same initial state, which means a delta function as the initial condition of the Fokker-Planck equation
- Consider the evolution between two resonance zones
- Let's call v_{exit} the velocity when the cavity exits the first resonance zone
- Calculate the final velocity when it reaches the second resonance zone or it come back to the first
- Do it with and without an applied force

Numerical simulation

$$\tau_1 = 0.04\mathrm{s}$$
$$v_{exit} = p\omega_0\sigma_\infty$$
$$F_0 = -8 \times 10^{-6}\mathrm{ms}^{-1}$$
$$F_0 = 0$$

Cumulative probability distribution of final velocity in units of v_{exit}



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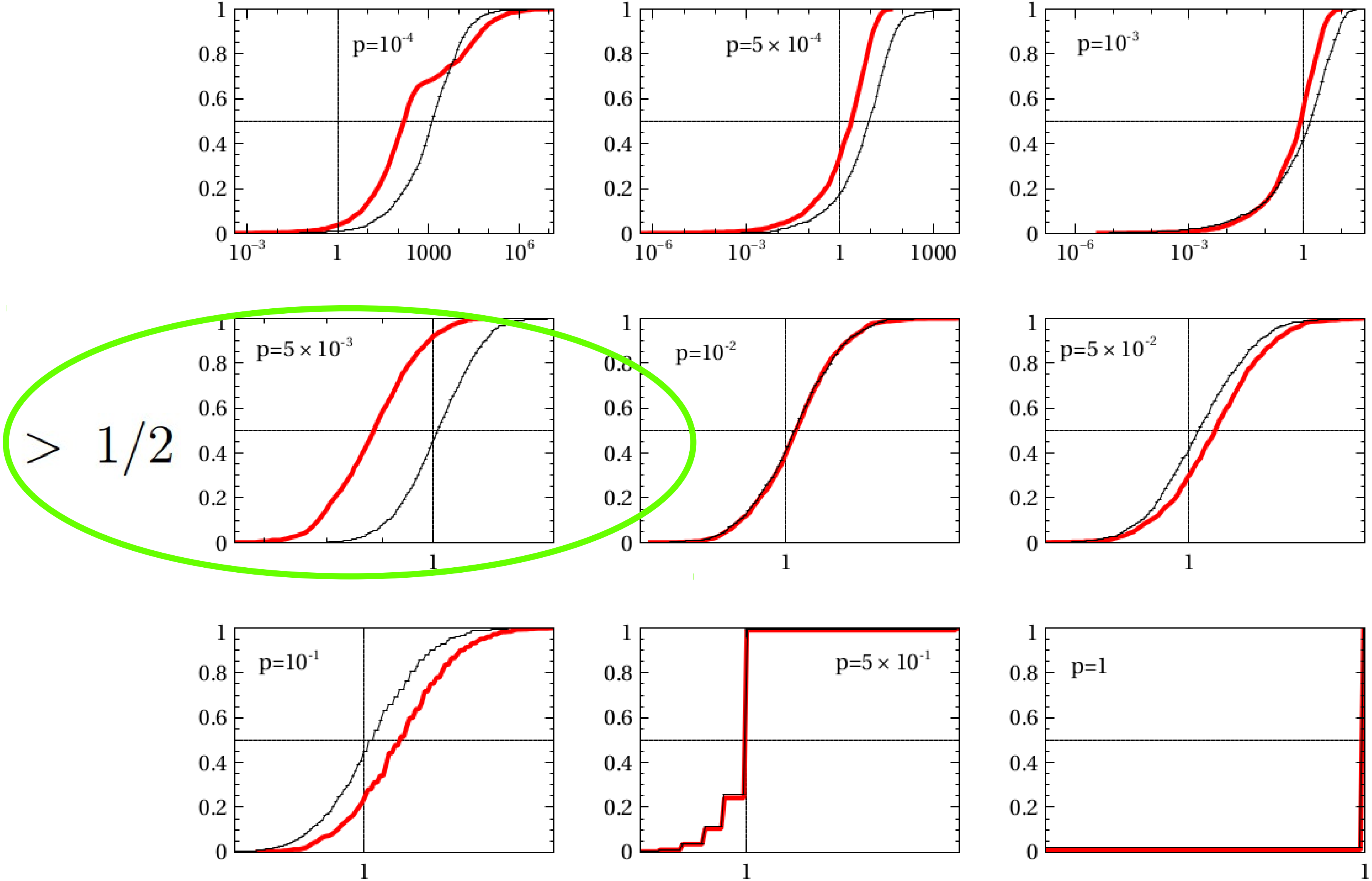


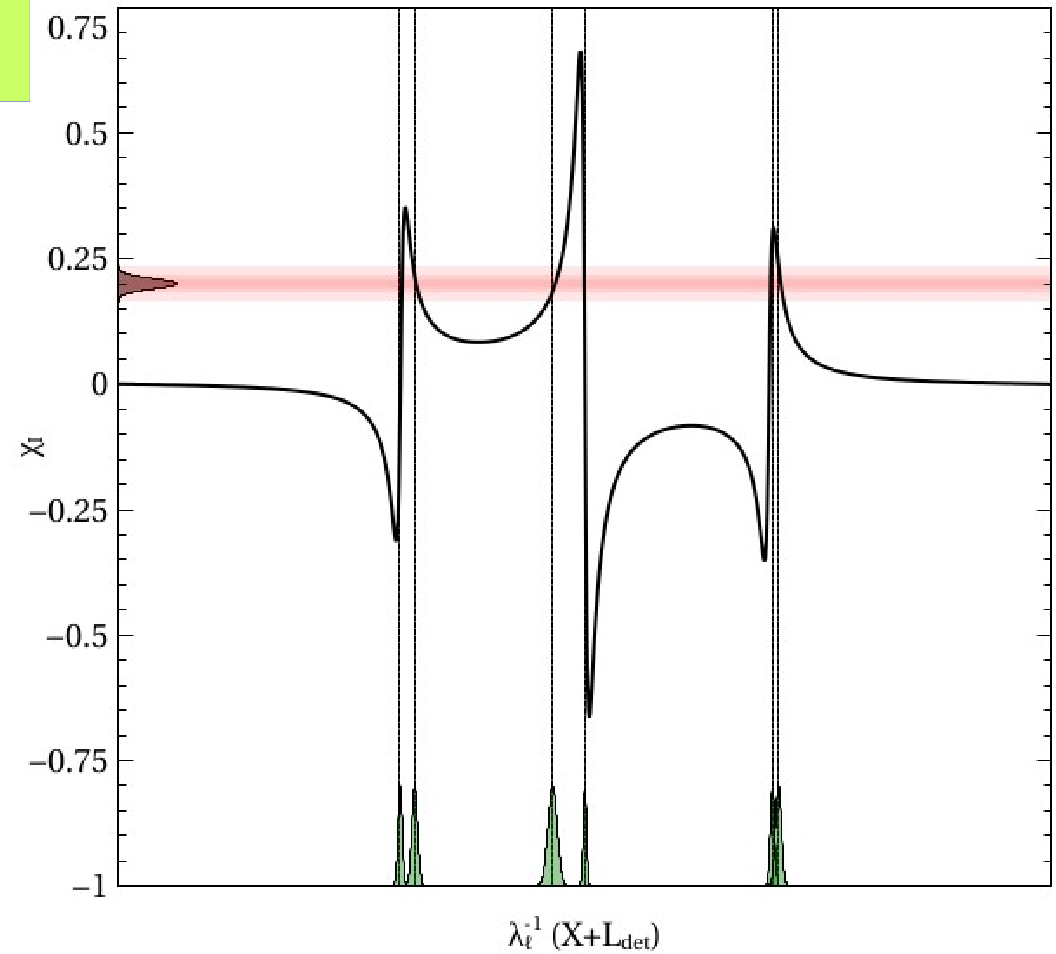
Further work

- For each value of the parameter τ_1 we have to find the velocity with the maximum probability of getting the cavity slowed down.
- This simulation was only the first step of Bayesian approach: the evolution of the probability distribution.
- Next we will implement also the update step.

Improving the model

- Include PDH information
- Include ring effects in the PDH signal
- Include radiation pressure in the dynamics

→ **non gaussian probability distributions**



How to parametrize them?

Sum of gaussians

$$\sum \omega_i \mathcal{N}(\mu_i, \mathbb{C})$$

→

Evolution of gaussians

$$\sum \omega_i \mathcal{N}(\mu_i^*, \mathbb{C}^*)$$

→

Weight update

$$\sum \omega_i^* \mathcal{N}(\mu_i^*, \mathbb{C}^*)$$

$\underbrace{\hspace{10em}}_{\text{Particle filter}}$

Particle filter

Include dynamics parameters

$$x^{ext} = \begin{pmatrix} x \\ p \end{pmatrix}$$

SIMPLE EXAMPLE

$$\dot{p} = -m\omega^2 x$$

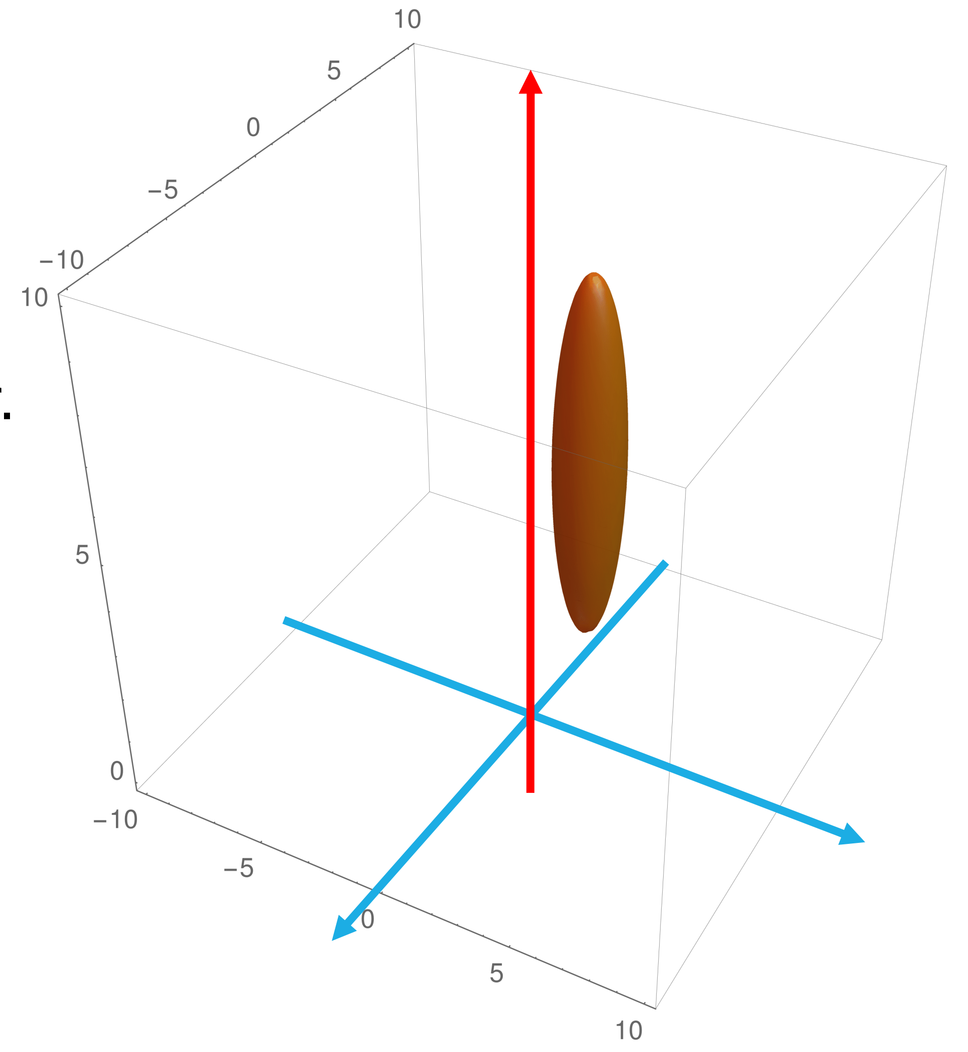
$$\dot{x} = \frac{1}{m} p$$

- We measure the state (position and velocity). With some measurement error.

- We enlarge the space, adding the unknown parameter

- We model our ignorance with a joint probability distribution

- We assume we have a good model…



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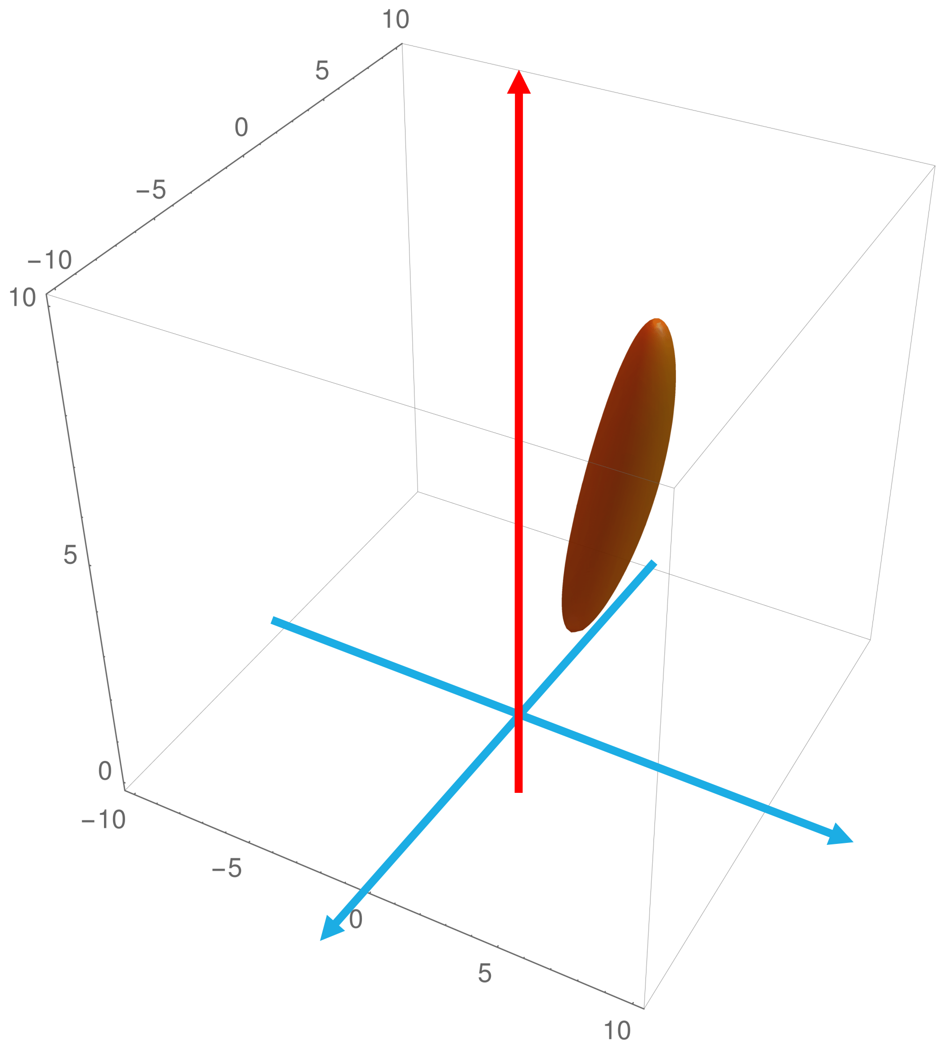

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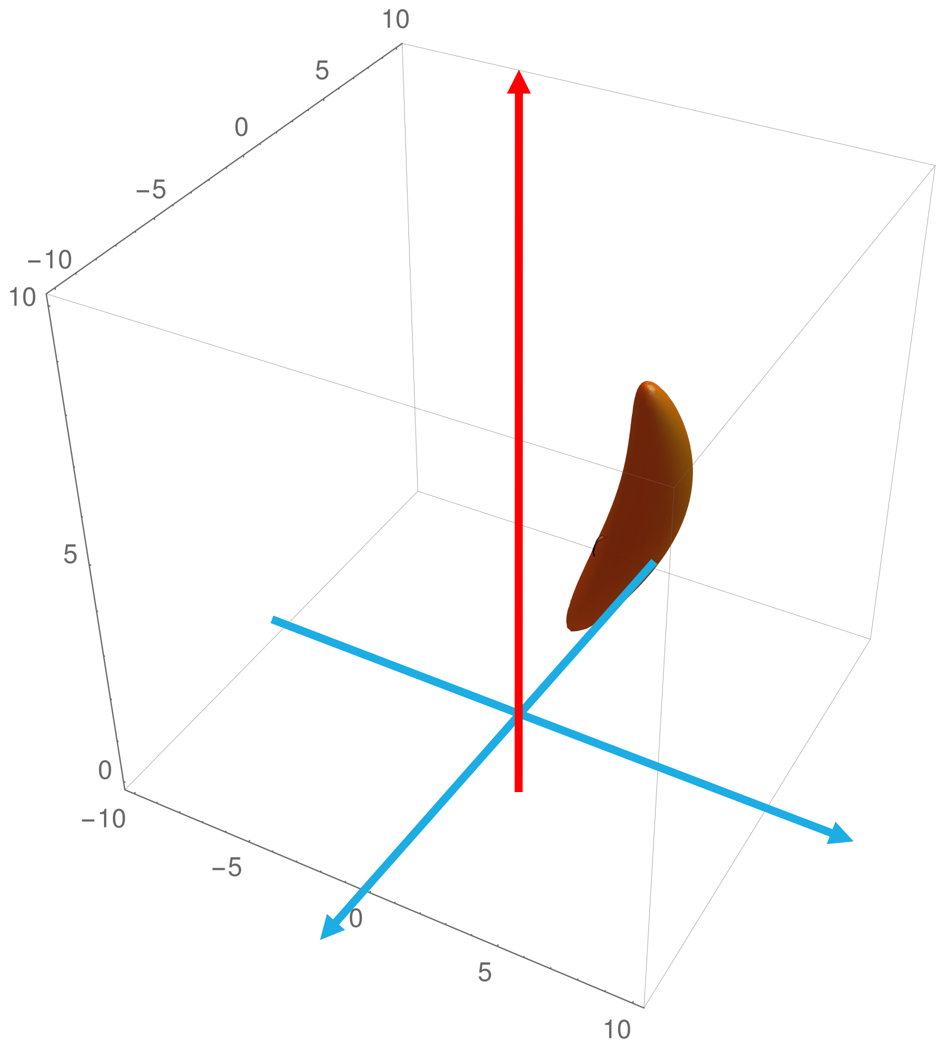

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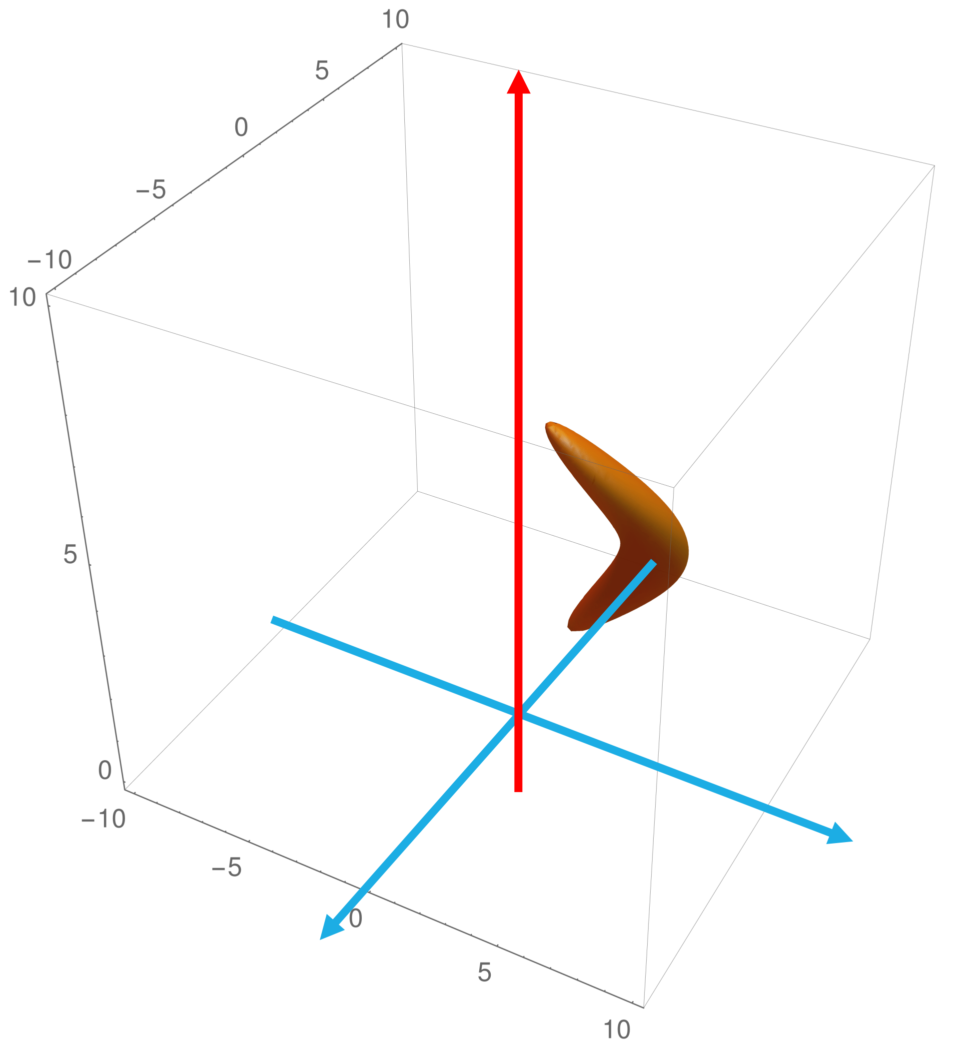

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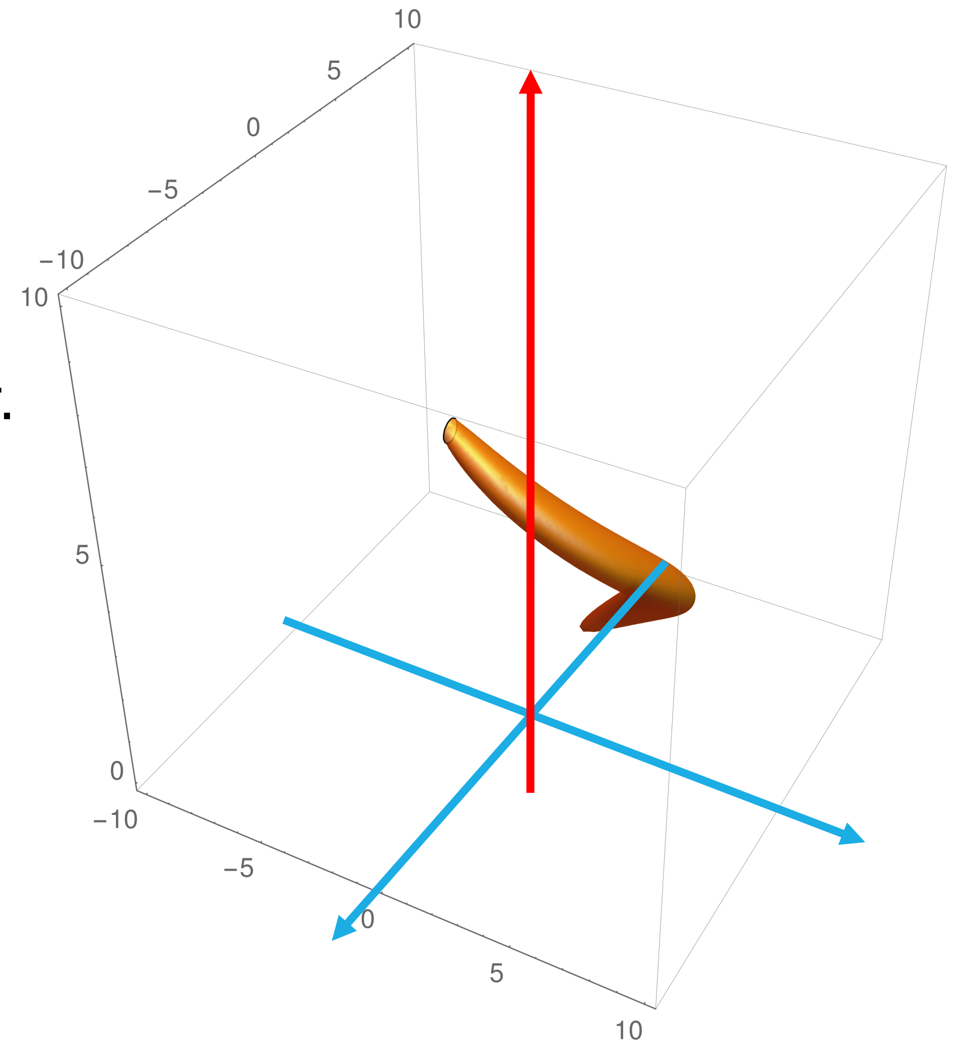

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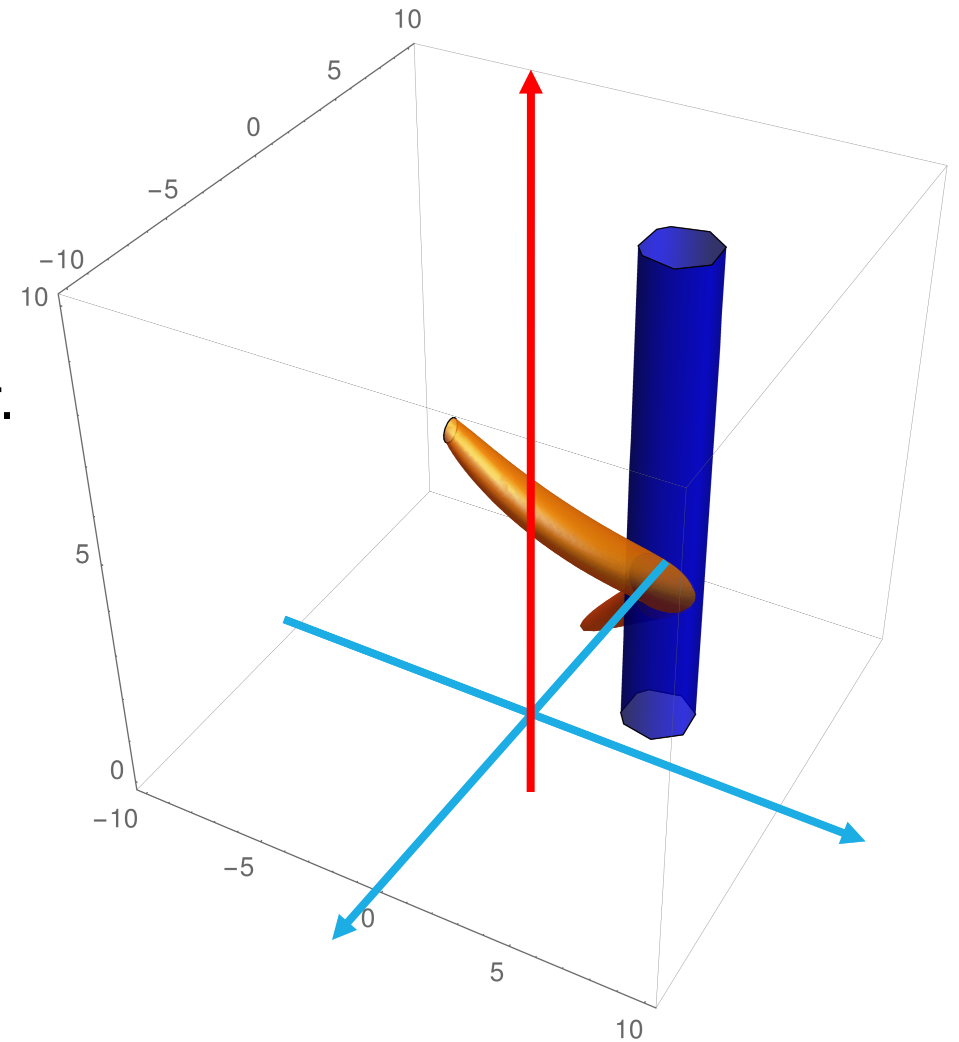
- Now, we measure the position and the velocity again,
and apply the Bayesian update step



Thank you for the attention