Thermo-optic noise at cryogenic temperature

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1 Introduction

Thermal noise is a fundamental limit of precious measurement, for example, gravitational wave detection, frequency stabilization with rigid cavity and so on. Thermo-optic noise [1] is a kind of thermal noise. It has already been studied well in the case of room temperature. Nevertheless, this noise at cryogenic temperature was not investgated well. Since cryogenic technique is a popular one to reduce thernal noise in precious measurment, thermo-optic noise at cryogenic temperature must be investigated. There are two problems. The first one is frequency dependence of power spectral density at cryogenice temperature is different from that at room temperature because material thermal properties of substrate drastically changes when the mirror is cooled. The second point is that we do not know the material thermal properties of coating at cryogenic temperature well.

In this report, I will show the solutions for these problems. The frequency dependence of power spectrum at cryogenic temperature is calculated. The method to evaluate constrains on thermal properties of coating from measured spectrum is explained.

2 Outline of thermo-optic noise

Thermo-optic noise is the summation of coating thermoelastic noise [2, 3] and thermo-refractive noise [4]. Both of them are caused by the temperature fluctuation in coating. The coating thermoelasic noise is the fluctuation of the surface of the coating (boundary between coating and vacuum). The light is not reflected on this surface perfectly and able to go into coating slightly. The fluctuation of this optical thickness is called thermo-refractive noise. When we calculate thermo-optic noise, we must take correlation between coating thremoelasitc noise and thermo-refractive noise because they have the common origin [1].

3 Coating thermoelastic noise

The method developed by Levin [5] is one of the most popular ones to calculate mirror thermal noise. When we calculate the fluctuation of surface, the pressure of which profile is the same as that of laser beam (in this paper, it is Gaussian) is applied on mirror surface. The mechanical dissipation caused by this pressure is related to thermal noise by the fluctuation-dissipation theorem.

When the pressure is applied, the mirror deforms. Since the mirror substrate has thermal expansion coefficient, the temperature gradient appears. The relaxation of this gradient is the mechanical loss called thermoelastic damping. This is related to (substrate) thermoelastic noise [6, 7]. It is possible to derive the thermoelastic noise from equation of motion of elastic body and heat condution equation.

Coating thermoelastic noise stems from the relaxation of temperature difference between susbstrate and coating due to discrepancy of meterial between substrate and coating. The relaxation of temperature difference must be evaluated. In the case of room temperature, only the heat flux which is perpendicular to mirror flat surface is taken into account [2, 3]. Because the relaxation time along radius direction is longer than period of targeted gravitational wave (1 msec $\sim$ 100 msec). However, this assumtion is not appropriate at cryogenic temperature. Cooled mirror substrate (sapphire or silicon) has large thermal conductivity and small specific heat. Thus, heat flux along radius direction should be considered.

Here, details of calculation are introduced. We adopt cyrindrical coordinate (r,θ,z). The z-axis is the same as the optical axis. On the boundary between substrate and coating, z is equal to zero. The Gaussian pressure is applied on the boundary between vacuum and coating. The profile is descrived as

$$P(r) = F\frac{2}{\pi {r_0}^2}\exp\left(-\frac{2r^2}{{r_0}^2}\right), \tag{1}$$

where r_0 is beam radius, F is the amplitude of pressure. The laser beam should be smaller than mirror itself. It is assumed that the mirror is a half-infinite elastic body ($z > 0$). The deformation of substrate is the same as that without coating because coating is too thin. Since we are interested with thermal noise below fundamental mode of mirror, it is suppoed that the pressure is static when equation of motion of elastic body is solved. This solution is well known. The strain components in substrate near coating are as follows [8].

$$u_{rr} = \frac{F}{2\pi}\frac{(1+\sigma_0)(1-2\sigma_0)}{E_0}\left[\frac{1}{r^2}\left(1-\exp(-\frac{2r^2}{{r_0}^2})\right)-\frac{4}{{r_0}^2}\exp(-\frac{2r^2}{{r_0}^2})\right], \tag{2}$$

$$u_{\theta\theta} = -\frac{F}{2\pi}\frac{(1+\sigma_0)(1-2\sigma_0)}{E_0}\frac{1}{r^2}\left(1-\exp(-\frac{2r^2}{{r_0}^2})\right), \tag{3}$$

$$u_{zz} = -\frac{F}{2\pi}\frac{(1+\sigma_0)(1-2\sigma_0)}{E_0}\frac{4}{{r_0}^2}\exp(-\frac{2r^2}{{r_0}^2}), \tag{4}$$

where E_0 and σ_0 are Young modulus and Poisson ratio of substrate. Since our system is axis symmetric, $u_{r\theta}$ and $u_{z\theta}$ are zero. Moreover, the pressure is parallel to z-axis, u_{rz} is also zero. Since the coating is thin, strain along surface sholud follow that in substrate. On the other hand, stress along optical axis in coating should be the same as that in substrate. The strain components in coating are as follows [8];

$$u_{rr} = \frac{F}{2\pi}\frac{(1+\sigma_0)(1-2\sigma_0)}{E_0}\left[\frac{1}{r^2}\left(1-\exp(-\frac{2r^2}{{r_0}^2})\right)-\frac{4}{{r_0}^2}\exp(-\frac{2r^2}{{r_0}^2})\right], \tag{5}$$

$$u_{\theta\theta} = -\frac{F}{2\pi}\frac{(1+\sigma_0)(1-2\sigma_0)}{E_0}\frac{1}{r^2}\left(1-\exp(-\frac{2r^2}{{r_0}^2})\right), \tag{6}$$

$$u_{zz} = -\frac{F}{2\pi}\frac{(1+\sigma_0)(1-2\sigma_0)}{E_0}\frac{(1+\sigma_1)(1-2\sigma_1)}{E_1(1-\sigma_1)}\left(\frac{E_0}{(1+\sigma_0)(1-2\sigma_0)}-\frac{E_1\sigma_1}{(1+\sigma_1)(1-2\sigma_1)}\right)\frac{4}{{r_0}^2}\exp(-\frac{2r^2}{{r_0}^2}) \tag{7}$$

where E_1 and σ_1 are Young modulus and Poisson ratio of coating.

Above strain components generate temperauter gradient and difference. The heat conduction equation tells us how these gradient and difference relaxies. Before this equation is solved, outlines are considered. The coating consists of two different material, SiO_2 and Ta_2O_5. Since each layer is less than half of 1 μm, time scale of heat exchange between SiO_2 and Ta_2O_5 layers is shorter than period of gravitational wave observation band. At room temperature, the relaxation time is 0.2 μsec [3]. If the mirror is cooled, this relaxation time becomes shorter. Therefore, in all published papers[2, 3], it is assumed that temperature in coating is homogeneous. Next problem is the time scale of heat exchange of coating and substrate. Even in the case of room temperature, this time scale is about 0.2 msec. It is higher than gravitational wave frequency. It implies that almost all heat generated in coating flows into substrate. Thus, coating itself can be treated as heat source on surface without thickness in this report. It must be noted that spectrum in high frequency region is evaluated in Refs. [3, 1].

In general form, the heat conduction equation is described as

$$\frac{\partial}{\partial t}\delta T-\frac{\kappa}{\rho C}\Delta(\delta T)=-\frac{\alpha ET}{\rho C(1-2\sigma)}\frac{\partial}{\partial t}(u_{rr}+u_{\theta\theta}+u_{zz}), \tag{8}$$

where δT is the temperature difference caused by strain. The quantities $\alpha, \kappa, \rho, C, T$ are the thermal expansion, thermal conductivity, density, specific heat per unit mass, temperature, respectively. The first and second terms of left hand side are change of temperature and heat diffusion. The right hand side is the generation of heat by strain. In our case, this equation is rewritten in frequency region as follows (in order to simplify the discussion, it is supposed that coating is homogeneous material);

$$\mathrm{i}\omega\delta\tilde{T}-\frac{\kappa_0}{\rho_0 C_0}\Delta(\delta\tilde{T})=-\mathrm{i}\omega\delta\tilde{T}_{\mathrm{sc}}d\delta(z)\frac{\rho_1 C_1}{\rho_0 C_0}, \tag{9}$$

where $d, \delta(z)$ and δT_{sc} are the coating thickness, Dirac delta function, and temperature difference between substrate and coating. Subscripts 0 and 1 imply substrate and coating.

It must be noted that this equation shows what happens in substrate, not coating. All parameters of coating are included in left hand side. Let us consider temperature difference $\delta T_{\rm sc}$ between substrate and coating. The temperature change in substrate near coating is derived from right hand side of Eq. (8) and strain, Eqs.(2),(3),(4).

$$\frac{\alpha_0 E_0 T}{\rho_0 C_0 (1-2\sigma_0)} \frac{F}{\pi} \frac{(1+\sigma_0)(1-2\sigma_0)}{E_0} \frac{4}{{r_0}^2} \exp(-\frac{2r^2}{{r_0}^2}), \tag{10}$$

The temperature change in coating is derived from strain, Eqs.(5),(6),(7).

$$\frac{\alpha_1 E_1 T}{\rho_1 C_1 (1-2\sigma_1)} \frac{F}{2\pi} \frac{(1+\sigma_0)(1-2\sigma_0)}{E_0} \left(\frac{E_0}{E_1} \frac{(1+\sigma_1)(1-2\sigma_1)}{(1+\sigma_0)(1-2\sigma_0)(1-\sigma_1)} + \frac{1-2\sigma_1}{1-\sigma_1} \right) \frac{4}{{r_0}^2} \exp(-\frac{2r^2}{{r_0}^2}), \tag{11}$$

Therefore, the temperature difference between substrate and coating is as follows;

$$\begin{aligned} \delta T_{\rm sc} &= \left[\frac{\alpha_1 E_1 T}{\rho_1 C_1 (1-2\sigma_1)} \frac{1}{2} \left(\frac{E_0}{E_1} \frac{(1+\sigma_1)(1-2\sigma_1)}{(1+\sigma_0)(1-2\sigma_0)(1-\sigma_1)} + \frac{1-2\sigma_1}{1-\sigma_1} \right) - \frac{\alpha_0 E_0 T}{\rho_0 C_0 (1-2\sigma_0)} \right] \\ &\times \frac{F}{\pi} \frac{(1+\sigma_0)(1-2\sigma_0)}{E_0} \frac{4}{{r_0}^2} \exp(-\frac{2r^2}{{r_0}^2}), \end{aligned} \tag{12}$$

The heat conduction equation Eq. (9) is rewritten as

$$\begin{aligned} {\rm i}\omega\tilde{\delta T} - \frac{\kappa_0}{\rho_0 C_0} \Delta(\tilde{\delta T}) &= -{\rm i}\omega \frac{\alpha_{\rm eff} E_0 T}{\rho_0 C_0 (1-2\sigma_0)} d\delta(z) \frac{F}{\pi} \frac{(1+\sigma_0)(1-2\sigma_0)}{E_0} \frac{4}{{r_0}^2} \exp(-\frac{2r^2}{{r_0}^2}) \qquad (13) \\ \alpha_{\rm eff} &= \alpha_1 \frac{1}{2} \left(\frac{(1+\sigma_1)}{(1+\sigma_0)(1-\sigma_1)} + \frac{E_1}{E_0} \frac{1-2\sigma_0}{1-\sigma_1} \right) - \alpha_0 \frac{\rho_1 C_1}{\rho_0 C_0} \qquad (14) \end{aligned}$$

All parameters of coating appear in only effective thermal expansion[1] $\alpha_{\rm eff}$. Here we remember coating consists of two kinds of material. In Eq. (13), coating is a heat source without thickness. Therefore, coating in total as heat source is the summation of that of SiO_2 and Ta_2O_5. The effective thermal expansion coefficient is rewritten as

$$\begin{aligned} \alpha_{\rm eff} &= \alpha_{\rm SiO2} \frac{d_{\rm SiO2}}{d} \frac{1}{2} \left(\frac{(1+\sigma_{\rm SiO2})}{(1+\sigma_0)(1-\sigma_{\rm SiO2})} + \frac{E_{\rm SiO2}}{E_0} \frac{1-2\sigma_0}{1-\sigma_{\rm SiO2}} \right) \\ &+ \alpha_{\rm Ta2O5} \frac{d_{\rm Ta2O5}}{d} \frac{1}{2} \left(\frac{(1+\sigma_{\rm Ta2O5})}{(1+\sigma_0)(1-\sigma_{\rm Ta2O5})} + \frac{E_{\rm Ta2O5}}{E_0} \frac{1-2\sigma_0}{1-\sigma_{\rm Ta2O5}} \right) \\ &- \alpha_0 \frac{d_{\rm SiO2}}{d} \frac{\rho_{\rm SiO2} C_{\rm SiO2}}{\rho_0 C_0} - \alpha_0 \frac{d_{\rm Ta2O5}}{d} \frac{\rho_{\rm Ta2O5} C_{\rm Ta2O5}}{\rho_0 C_0} \end{aligned} \tag{15}$$

Here, Eq. (13) is solved. On $z = 0$, heat flow must vanish (temperature gradient is zero). In order to fulfill this boundary condition, it is assumed that $\delta T(z)$ is even function [7]. Thus, the right hand side of Eq. (13) should be twice [2]. The solution is as follows.

$$\begin{aligned} \tilde{\delta T} &= \frac{4\alpha_{\rm eff} T dF(1+\sigma_0)}{\rho_0 C_0} \int \frac{-{\rm i}\omega}{{\rm i}\omega - \frac{\kappa_0}{\rho_0 C_0}({k_x}^2 + {k_y}^2 + {k_z}^2)} \exp\left(-\frac{({k_x}^2 + {k_y}^2){r_0}^2}{8} \right) \\ &\times \exp\left({\rm i}(k_x x + k_y y + k_z z) \right) \frac{dk_x dk_y dk_z}{(2\pi)^3}, \end{aligned} \tag{16}$$

[1]This effective thermal expansion coefficient in Ref. [2] is wrong. That in Ref. [3] is correct.

The dissipated power $W_{\rm diss}$ is described as

$$W_{\rm diss} = \frac{1}{2}\frac{\kappa_0}{2T}\int(\nabla\delta T)^2 dV \tag{17}$$

$$= \frac{1}{2}\frac{\kappa_0}{2T}\left(\frac{4\alpha_{\rm eff}TdF(1+\sigma_0)}{\rho_0 C_0}\right)^2 \times\int\frac{\omega^2({k_x}^2+{k_y}^2+{k_z}^2)}{\omega^2+(\frac{\kappa_0}{\rho_0 C_0})^2({k_x}^2+{k_y}^2+{k_z}^2)^2}\exp\left(-\frac{({k_x}^2+{k_y}^2){r_0}^2}{4}\right)\frac{dk_x dk_y dk_z}{(2\pi)^3} \tag{18}$$

$$= \frac{1}{2}\frac{\kappa_0}{T}\left(\frac{4\alpha_{\rm eff}TdF(1+\sigma_0)}{\rho_0 C_0}\right)^2\frac{\sqrt{2}}{\pi^2{r_0}^5}\int_0^\infty du\int_0^\infty dv\frac{\Omega^2 u(u^2+v^2)}{\Omega^2+(u^2+v^2)^2}\exp\left(-\frac{u^2}{2}\right) \tag{19}$$

$$= \frac{1}{2}\frac{T}{\kappa_0}\frac{\left(4\alpha_{\rm eff}dF(1+\sigma_0)\right)^2}{2\sqrt{2}\pi^2 r_0}\omega^2\int_0^\infty du\int_0^\infty dv\frac{u(u^2+v^2)}{\Omega^2+(u^2+v^2)^2}\exp\left(-\frac{u^2}{2}\right) \tag{20}$$

$$\Omega = \omega\left(\frac{\rho_0 C_0}{\kappa_0}\right)\left(\frac{{r_0}^2}{2}\right) \tag{21}$$

According to the fluctuation-dissipation theorem, power spectral density of coating thermoelastc noise $G_{\rm elastic}$ is written as

$$G_{\rm elastic} = \frac{8k_{\rm B}T}{\omega^2}\frac{W_{\rm diss}}{F^2} \tag{22}$$

$$= \frac{8k_{\rm B}T^2\left(\alpha_{\rm eff}d(1+\sigma_0)\right)^2}{\pi r_0\kappa_0}J(\Omega) \tag{23}$$

$$J(\Omega) = \frac{2\sqrt{2}}{\pi}\int_0^\infty du\int_0^\infty dv\frac{u(u^2+v^2)}{\Omega^2+(u^2+v^2)^2}\exp\left(-\frac{u^2}{2}\right) \tag{24}$$

Figure 1 shows $J(\Omega)$. If Ω is larger than unity, this function is described as

$$J(\Omega)\sim\frac{1}{\sqrt{\Omega}} \tag{25}$$

Therefore, in high frequny region, Eq. (23) is approximated as

$$G_{\rm elastic}\sim\frac{8\sqrt{2}k_{\rm B}T^2{\alpha_{\rm eff}}^2d^2(1+\sigma_0)^2}{\pi{r_0}^2\sqrt{\kappa_0\rho_0 C_0\omega}} \tag{26}$$

This formula is the same as that in Ref. [2][2]. If Ω is smaller than unity, J is $\sqrt{\pi}$. Therefore, in low frequency region, Eq. (23) is approximated as

$$G_{\rm elastic}\sim\frac{8k_{\rm B}T^2\left(\alpha_{\rm eff}d(1+\sigma_0)\right)^2}{\sqrt{\pi}r_0\kappa_0} \tag{27}$$

4 Thermo-refractive noise

Since the light can go into coating slightly, interferometer can detect the fluctuation of optical thickness of a part of coating. This is the thermo-refractive noise. In order to

[2]The definition of beam radius in Ref. [2] is $\sqrt{2}$ times larger than that in this report.

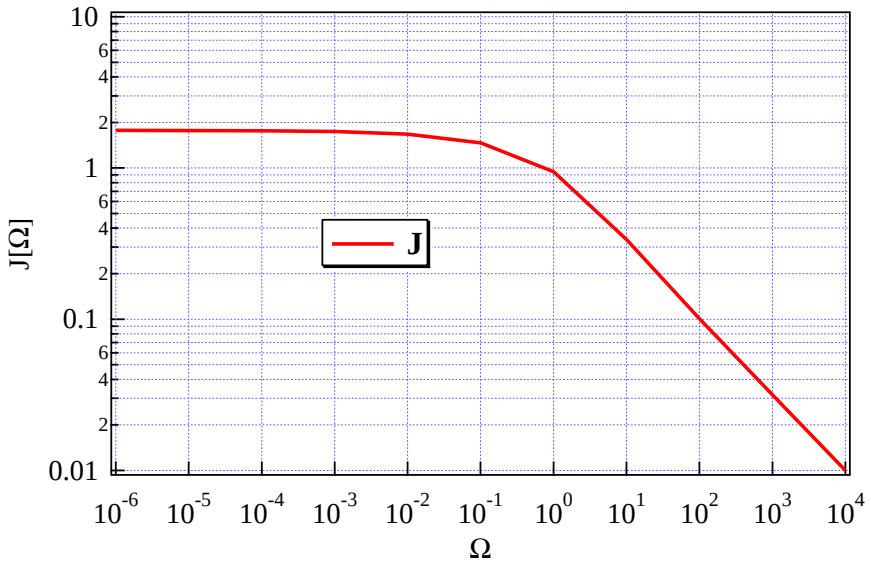

Figure 1: $J(\Omega)$ vs Ω

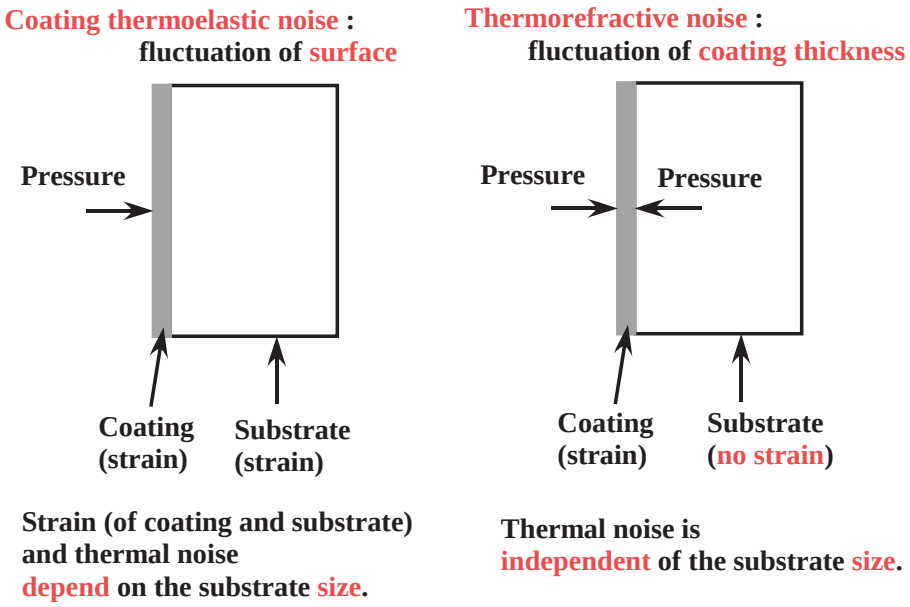

Figure 2: Difference between thermo-refractive noise and coating thermoelastic noise

calculate this optical thickness fluctuation, the pressure should be applied as like Fig. 2. In this case, substrate has no strain. Since the coating is too thin, effect of pressure of surface and that in coating cancel each other. Since the strain of coating which perpendicular to z-axis should follow those in substrate, only zz-component is not zero. In order to simplify the discussion, it is assumed that coating is uniform material. The strain in coating is as follows;

$$u_{zz} = -\frac{F}{2\pi}\frac{(1+\sigma_1)(1-2\sigma_1)}{E_1(1-\sigma_1)}\frac{4}{{r_0}^2}\exp(-\frac{2r^2}{{r_0}^2}). \tag{28}$$

As coating thermoelastic noise, coating is treated as heat source without thickness on mirror surface. Therefore, we can adopt Eq. (9). We must evaluate $\delta T_{\rm sc}$ in this case. This strain cause the heat via thermal expansion coefficient α_1 and temperature coefficient of refractive index β. Let us suppose that β is zero. The temperature change in coating is

$$\frac{\alpha_1 E_1 T}{\rho_1 C_1(1-2\sigma_1)}\frac{F}{\pi}\frac{(1+\sigma_1)(1-2\sigma_1)}{E_1(1-\sigma_1)}\frac{2}{{r_0}^2}\exp(-\frac{2r^2}{{r_0}^2}). \tag{29}$$

There is no temperature change in substrate because of no strain. Therefore, Eq. (9) is rewritten as

$$\mathrm{i}\omega\delta\tilde{T} - \frac{\kappa_0}{\rho_0 C_0}\Delta(\delta\tilde{T}) = -\mathrm{i}\omega\frac{\alpha_1 T}{\rho_0 C_0}\lambda\delta(z)\frac{F}{\pi}\frac{(1+\sigma_1)}{(1-\sigma_1)}\frac{2}{{r_0}^2}\exp(-\frac{2r^2}{{r_0}^2}), \tag{30}$$

where λ is wavelength. The light can go into coating slightly, not perfectly. Here, it is assumed that this depth is equal to the wavelength. Since Eq. (30) has the same style of Eq. (13), we must just rewrite Eq. (23) slightly. However, before rewriting, some items should be considered. At first, we must calculate the fluctuation of optical thickness, not geometrical thickness. Therefore, α_1 is replaced by $\alpha_1 n_1$, where n_1 is refrective index of coating. We must take β into account. In this case, Poisson ratio, which is the coupling between strain components along different direction, is zero. Thus, $\alpha_1 n_1(1+\sigma_1)/(1-\sigma_1)$ is rewritten as $\alpha_1 n_1(1+\sigma_1)/(1-\sigma_1)+\beta$. Finally, we must remember that coating consists of two kinds of material. The corrected $\alpha_1 n_1(1+\sigma_1)/(1-\sigma_1)+\beta$ is as follows [1][3].

$$\begin{aligned}\overline{\beta} &= \frac{{n_{\rm SiO2}}^2}{4({n_{\rm Ta2O5}}^2-{n_{\rm SiO2}}^2)}\left(\beta_{\rm Ta2O5}+\alpha_{\rm Ta2O5}n_{\rm Ta2O5}\frac{1+\sigma_{\rm Ta2O5}}{1-\sigma_{\rm Ta2O5}}\right)\\ &\quad+\frac{{n_{\rm Ta2O5}}^2}{4({n_{\rm Ta2O5}}^2-{n_{\rm SiO2}}^2)}\left(\beta_{\rm SiO2}+\alpha_{\rm SiO2}n_{\rm SiO2}\frac{1+\sigma_{\rm SiO2}}{1-\sigma_{\rm SiO2}}\right)\end{aligned} \tag{31}$$

Equation (30) is rewritten as

$$\mathrm{i}\omega\delta\tilde{T} - \frac{\kappa_0}{\rho_0 C_0}\Delta(\delta\tilde{T}) = -\mathrm{i}\omega\frac{\overline{\beta}T}{\rho_0 C_0}\lambda\delta(z)\frac{F}{\pi}\frac{2}{{r_0}^2}\exp(-\frac{2r^2}{{r_0}^2}), \tag{32}$$

[3]This correction is for quater wave doublets. If there is a half wave cap, other formula [1] is necessary.

Therefore, power specral density of thermo-refractive noise $G_{\rm refractive}$ is

$$G_{\rm refractive} = \frac{2k_{\rm B}T^2\left(\overline{\beta}\lambda\right)^2}{\pi r_0 \kappa_0} J(\Omega). \tag{33}$$

The high frequency appoxiamtion is

$$G_{\rm refractive} \sim \frac{2\sqrt{2}k_{\rm B}T^2\overline{\beta}^2\lambda^2}{\pi {r_0}^2\sqrt{\kappa_0\rho_0 C_0\omega}}. \tag{34}$$

This formula is the same as that in Ref. [4][4]. The low frequency approximation is

$$G_{\rm refractive} \sim \frac{2k_{\rm B}T^2\left(\overline{\beta}\lambda\right)^2}{\sqrt{\pi} r_0 \kappa_0} \tag{35}$$

The difference between results of this report and that of Ref. [1] must be noted. In this report, the pressure is applied as Fig. 2. On the other hand, in Ref. [1], averaged temperature fluctuation is evaluated. The results of these two method should be the same. However, if the method in Ref. [1] is adopted, some items must be taken into account. The coating thermoelastic noise is the fluctuation of surface. In this case, we must consider how temperature fluctuation is converted to surface fluctuation. The point is that coating expansion makes deformation of substrate. This deformation causes displacement of mirror surface. This backaction of substrate should be taken into account. The effective thermal expansion coefficient of coating thermoelastic noise is $2\alpha(1+\sigma)$. The reason why this value is 2α even if the Poisson ratio is zero is backaction of substrate. In the case of thermorefractive noise, we must evaluate fluctuation of thickness of coating. Since coating is extremly thin, the effect of backaction of substrate on both the side of coating is almost the same. Therefore, backaction of susbstrate is not able to change the thickness of coating. The effective expansion coefficient of thermo-refractive noise should be different from that of coating thermoelastic noise nevetheless they are the same in Ref. [1]. According to this report, effective coefficient of thermorefractive noise is $\alpha(1+\sigma)/(1-\sigma)$. This value becomes α if Poisson ratio vanishes.

5 Thermo-optic noise

When we calculate the thermo-optic noise, the pressure for coating thermoelastic noise and thermo-refractive noise are applied at the same times. Since ditribution of heat source, which is the left hand side of Eqs. (13) and (30), there is a perfect correlation between coating thermoelastic noise and thermo-refractive noise. The problem is whether the sign of correation is positive or negative. Let us assume that $\alpha_{\rm eff}$ and $\overline{\beta}$ are positive and that temperature increases slightly. Geomterical and optical thicknesses of coating

[4]The definition of beam radius in Ref. [2] is $\sqrt{2}$ times larger than that in this report.

also increase. Coating thermoelastic noise makes cavity length shorter. On the other hand, thermo-refractive noise makes cavity length longer. Thus, correlation is negative. Thermo-optics noise is expressed as

$$G_{\rm optic} = \frac{2k_{\rm B}T^2\left(2\alpha_{\rm eff}d(1+\sigma_0)-\overline{\beta}\lambda\right)^2}{\pi r_0\kappa_0}J(\Omega). \tag{36}$$

The high frequency appoxiamtion is

$$G_{\rm optic} \sim \frac{2\sqrt{2}k_{\rm B}T^2(2\alpha_{\rm eff}d(1+\sigma_0)-\overline{\beta}\lambda)^2}{\pi {r_0}^2\sqrt{\kappa_0\rho_0 C_0\omega}}. \tag{37}$$

The low frequency approximation is

$$G_{\rm optic} \sim \frac{2k_{\rm B}T^2\left(2\alpha_{\rm eff}d(1+\sigma_0)-\overline{\beta}\lambda\right)^2}{\sqrt{\pi}r_0\kappa_0}. \tag{38}$$

6 Constrain on coating properties

Although we obtain the formula of thermo-optic noise at cryogenic temperature, the other problem remains. We do not know the material properties of coating (especially, at cryogenic temperature) well. Here, it is consider how to put constrain on coating properties besed on the measured spectrum and formula of thermo-optic noise.

Except for $\alpha_{\rm eff}$ and $\overline{\beta}$, the paremeters in thermo-optic noise formula Eq. (36) is independent of coating. Therefore, if the measured spectrum is compared with Eq.(36), we obtain the upper limit of absolute value of $2(1+\sigma_0)\alpha_{\rm eff}d+\overline{\beta}\lambda$. Although this value depends on many parameters about coating, we assume that we know elastic property and density. They are almost independent of temperature and are on the same order of magnitude in usual cases. Therefore, remained problem is thermal expansion, temperature coefficient of refractive index, specific heat. It must be noted that $\alpha_{\rm eff}$ depends on substrate. On the other hand, $\overline{\beta}$ is independent of substrate. If we measure the spectrum two kinds of mirror, different substrate with same coating, we can obtain the upper limits of thermal expansion coefficient and $\overline{\beta}$ separately.

6.1 Constrain at room temperature

We can use approximated formula Eq. (37). Fortunately, the excellent sensitivity has already obatained. In the case of fused silica mirror, there are two kinds of excellent apparatus; gravitational wave detector and thermal noise interferometer. Their sensitivity and upper limit of $2(1+\sigma_0)\alpha_{\rm eff}d+\overline{\beta}\lambda$ are as follows.

- LIGO[9][5]

[5]Beam radius is 36 mm and 45 mm.

- Measured power spectrum: 1×10^{-19} m/$\sqrt{\text{Hz}}$ at 100 Hz
- Upper limit : $|2(1+\sigma_0)\alpha_{\text{eff}} d + \overline{\beta}\lambda| < 6.4 \times 10^{-10}$ m/K

- Thermal noise interferometer (University of Tokyo [10][6])
  - Measured power spectrum: 4×10^{-18} m/$\sqrt{\text{Hz}}$ at 1 kHz
  - Upper limit : $|2(1+\sigma_0)\alpha_{\text{eff}} d + \overline{\beta}\lambda| < 3.8 \times 10^{-11}$ m/K

The sensitivity of gravitational wave is better. However, the constrain of thermal noise interferometer is smaller. This is because the beam radius in thermal noise interferometer is smaller. The excellent spectra with sapphire mirror are also obtained.

- CLIO[12][7]
  - Measured power spectrum: 2×10^{-19} m/$\sqrt{\text{Hz}}$ at 300 Hz
  - Upper limit : $|2(1+\sigma_0)\alpha_{\text{eff}} d + \overline{\beta}\lambda| < 4.0 \times 10^{-10}$ m/K

- Thermal noise interferometer (Caltech [13][8])
  - Measured power spectrum: 4×10^{-18} m/$\sqrt{\text{Hz}}$ at 1 kHz
  - Upper limit : $|2(1+\sigma_0)\alpha_{\text{eff}} d + \overline{\beta}\lambda| < 2.1 \times 10^{-10}$ m/K

Therefore, we adopt the upper limit of thermal noise interferometer with fused silica and sapphire mirrors.

The effective thermal expansion coefficients α_{eff} with fused silica and sapphire substrate are considered. Equation (15) is rewritten as

$$\alpha_{\text{eff(FS)}} = 0.59\alpha_{\text{SiO}} + 0.63\alpha_{\text{TaO}} - 5.6 \times 10^{-7}[/\text{K}](\text{fusedsilica}) \quad (39)$$

$$\alpha_{\text{eff(Sa)}} = 0.35\alpha_{\text{SiO}} + 0.30\alpha_{\text{TaO}} - 3.0 \times 10^{-6}[/\text{K}](\text{sapphire}) \quad (40)$$

The parameters in Table 1 are used. Fortunately, specific heat at room temperature is known. In order to simplify the discussion, these effective coefficients are rewritten as

$$\alpha_{\text{eff(FS)}} \sim 1.2\alpha_{\text{ave}} - 5.6 \times 10^{-7}[/\text{K}](\text{fusedsilica}) \quad (41)$$

$$\alpha_{\text{eff(Sa)}} \sim 0.65\alpha_{\text{ave}} - 3.0 \times 10^{-6}[/\text{K}](\text{sapphire}) \quad (42)$$

$$\alpha_{\text{ave}} = \frac{\alpha_{\text{SiO}} + \alpha_{\text{TaO}}}{2} \quad (43)$$

Therefore, the upper limits derived from thermal noise interferometer are described as

$$|1.3 * 10^{-5}[\text{m}]\alpha_{\text{ave}} - 6.2 \times 10^{-12}[\text{m/K}] - 1.0 \times 10^{-6}[\text{m}]\overline{\beta}| < 3.8 \times 10^{-11} \text{ m/K}(\text{fusedsilica}) \quad (44)$$

$$|7.9 \times 10^{-6}[\text{m}]\alpha_{\text{ave}} - 3.6 \times 10^{-11}[\text{m/K}] - 1.0 \times 10^{-6}[\text{m}]\overline{\beta}| < 2.1 \times 10^{-10} \text{ m/K}(\text{sapphire}) \quad (45)$$

[6]Beam radius is 49 μm and 85 μm. Caltech group also obtained similar spectrum[11].

[7]Beam radius is 4.9 mm and 8.5 mm.

[8]Beam radius is 160 μm and 160 μm.

Table 1: Mechanical properties of mirror materials

Material	Sapphire	Silicon	Calcium fluoride	Fused silica	Ta_2O_5
Young's modulus [10^{10} Pa]	40	16	11	7.2	14
Poisson ratio	0.29	0.22	0.28	0.17	0.23
Density [g/cm^3]	4	2.33	3.18	2.2	—

Figure 3 shows this region. We obtain the region of thermal expansion coefficient -5×10^{-5} /K $< \alpha_{\rm ave} < 4 \times 10^{-5}$ /K and temperature coefficient of refractive index -7×10^{-4} /K$\overline{\beta} < 5 \times 10^{-4}$ /K. These upper limits are larger than the typical values. However, they are not so unrealistic. It must be repeated that measurement of two kinds of mirrors (different substrate) provides each upper limit of $\alpha_{\rm ave}$ and $\overline{\beta}$. The difference of slope is caused by the difference of substrate. The error of specific heat of coating is not serious. This contribution of specific heat is smaller than that of measured spectrum, which is proportional to width of region in Fig. 3.

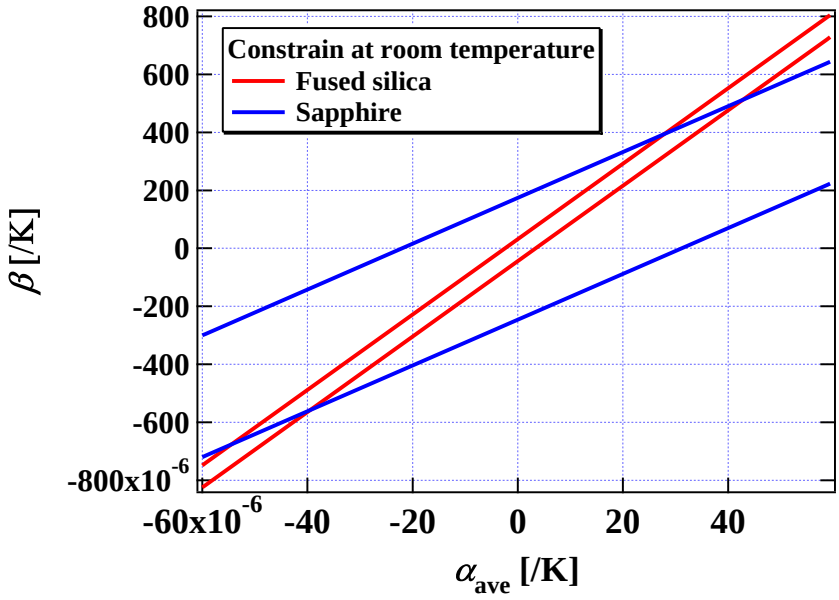

Figure 3: Constrain at room temperature. Outside of the region between two red (blue) lines is excluded by measured spectrum of thermal noise interferoemeter with fused silica (sapphire) mirrors.

6.2 Constrain at cryogenic temperature

The parameters $\alpha_{\rm ave}, \overline{\beta}, C_{\rm SiO}, C_{\rm TaO}$ depend on temperature. However, temperature depepndence of specific heat is known. It is similar to that of crystal, which is proportional to cubic of temperature. Therefore, we are able to neglect contribution of specific heat when we discuss the upper limit of $\alpha_{\rm ave}$ and $\overline{\beta}$. The sapphire and silicon are candidate for substrate of cryogenic mirror. If the substrate is silicon, $\alpha_{\rm eff}$ is described as

$$\begin{aligned} \alpha_{\rm eff(Si)} &= 0.43\alpha_{\rm SiO} + 0.40\alpha_{\rm TaO}({\rm silicon}) && (46) \\ &\sim 0.82\alpha_{\rm ave} && (47) \end{aligned}$$

It is expected that fluctuation of cryogenic cavity is limited by coating Brownian noise. It is assumed that it is true. The upper limits of $\alpha_{\rm ave}$ and $\overline{\beta}$ are considered in such a case. The coating Brownina noise are summarized here. The beam radii are 1 mm.

- Sapphire
  - 20 K: 6×10^{-20} m/$\sqrt{\rm Hz}$ at 1 kHz
  - 4 K: 3×10^{-20} m/$\sqrt{\rm Hz}$ at 1 kHz
- Silicon
  - 20 K: 10×10^{-20} m/$\sqrt{\rm Hz}$ at 1 kHz
  - 4 K: 5×10^{-20} m/$\sqrt{\rm Hz}$ at 1 kHz

We are able to use approximated formula Eq. (38). The upper limits are as follows.

- Sapphire
  - 20 K: $|2(1+\sigma_0)\alpha_{\rm eff}d + \overline{\beta}\lambda| < 1.6 \times 10^{-9}$ m/K
  - 4 K: $|2(1+\sigma_0)\alpha_{\rm eff}d + \overline{\beta}\lambda| < 4 \times 10^{-10}$ m/K
- Silicon
  - 20 K: $|2(1+\sigma_0)\alpha_{\rm eff}d + \overline{\beta}\lambda| < 1.4 \times 10^{-9}$ m/K
  - 4 K: $|2(1+\sigma_0)\alpha_{\rm eff}d + \overline{\beta}\lambda| < 7.5 \times 10^{-10}$ m/K

These upper limits are rewritten as

- Sapphire
  - 20 K: $|7.9 \times 10^{-6}[{\rm m}]\alpha_{\rm ave} - 1.0 \times 10^{-6}[{\rm m}]\overline{\beta}| < 1.6 \times 10^{-9}$ m/K
  - 4 K: $|7.9 \times 10^{-6}[{\rm m}]\alpha_{\rm ave} - 1.0 \times 10^{-6}[{\rm m}]\overline{\beta}| < 4 \times 10^{-10}$ m/K
- Silicon

– 20 K: $|9.4 \times 10^{-6}[\mathrm{m}]\alpha_{\mathrm{ave}} - 1.0 \times 10^{-6}[\mathrm{m}]\overline{\beta}| < 1.4 \times 10^{-9}$ m/K

– 4 K: $|9.4 \times 10^{-6}[\mathrm{m}]\alpha_{\mathrm{ave}} - 1.0 \times 10^{-6}[\mathrm{m}]\overline{\beta}| < 7.5 \times 10^{-10}$ m/K

Figures 4 and 5 show these region. Unfortunatly, upper limit is not so small. Moreover, slope of sapphire is comparable with that of silicon. Thus, the upper limit of absolute values of α_{ave} (2×10^{-3} /K at 20 K, 7×10^{-4} /K at 4 K) and $\overline{\beta}$ (2×10^{-2} /K at 20 K, 6×10^{-3} /K at 4 K) is large. It is note that smaller beam radius provides larger upper limit. The linear spectra of coating Brownian noise and thermo-optic noise are inversely proportional to beam radius itself and square root of it, respectively. However, above discussion implies that thermo-optic noise is not so serious at cryogenic temperature. Under the same assumption, the upper limit of linear spectram densities of thermo-optic noise of CLIO and LCGT are calculated. In both interfetometers, mirror are made from sapphire. Temperature is 20 K. The upper limit of thermo-optic noise in CLIO and LCGT is 2.4×10^{-20} m/$\sqrt{\mathrm{Hz}}$ and 1.1×10^{-20} m/$\sqrt{\mathrm{Hz}}$. The goal sensitivity around 100 Hz is 3×10^{-20} m/$\sqrt{\mathrm{Hz}}$ and 1.2×10^{-20} m/$\sqrt{\mathrm{Hz}}$.

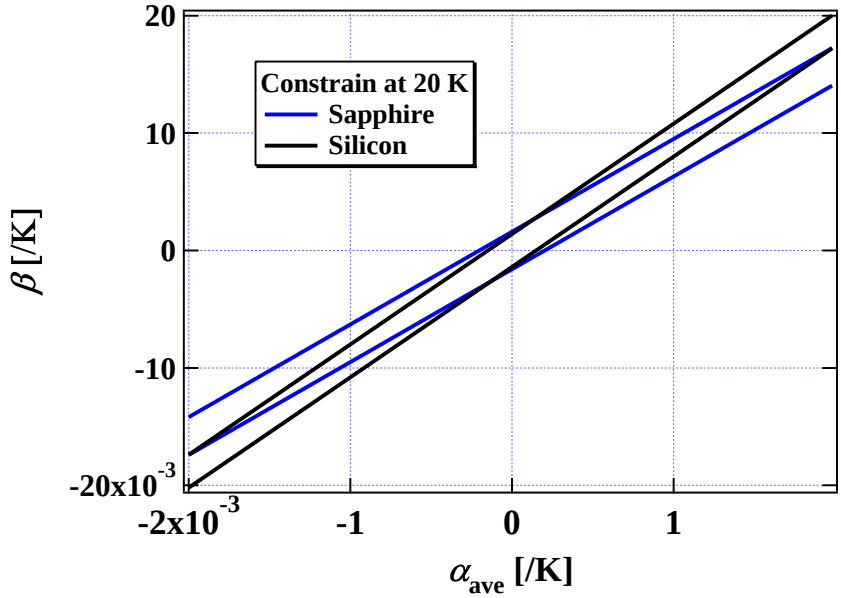

Figure 4: Constrain at 20 K. Outside of the region between two blue (black) lines is excluded by measured spectrum of thermal noise interferoemeter with sapphire (silicon) mirrors.

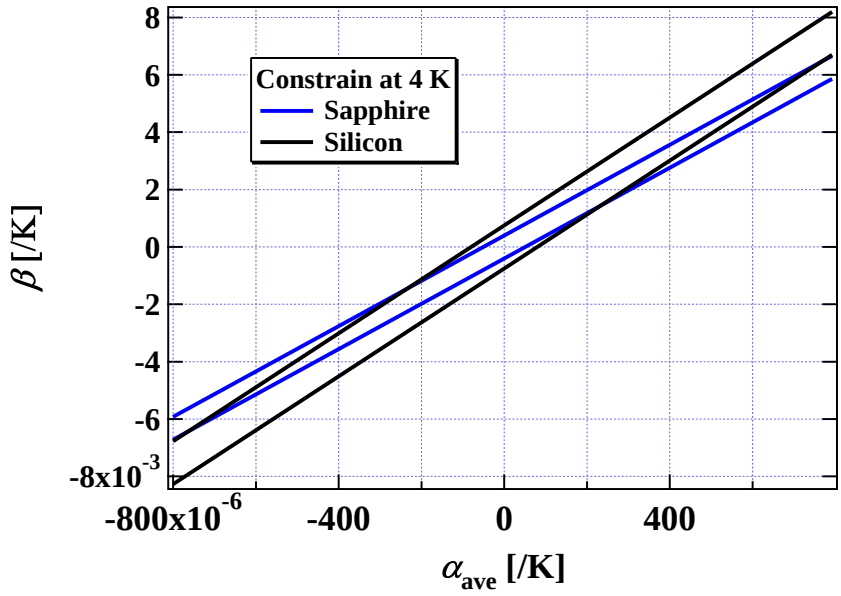

Figure 5: Constrain at 4 K. Outside of the region between two blue (black) lines is excluded by measured spectrum of thermal noise interferoemeter with fused sapphire (silicon) mirrors.

7 Summary

In this report, the formula of the thermo-optic noise at cryogenic temperature is derived. The formula at room temperature is not valid because substrate material properties drastically change when the mirror is cooled. We do not know the material properties of coating at cryogenic temperature. However, we are able to derive constrain of coating properties (thermal expansion $\alpha_{\rm ave}$ and temperature coefficient of refractive index $\overline{\beta}$) from the thermo-optic noise formula and measured power spectrum in thermal noise interferometer. If we obtain the spectra with two kinds of mirror (different material and same coating), we can derive limits of $\alpha_{\rm ave}$ and $\overline{\beta}$ separately. The upper limit at room temperautre is larger than typical value. However, they are not so unrealistic. The upper limit at cryogenic temperature is large. It implies that thermo-optic noise is not a serious problem in cryogenic experiments.

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