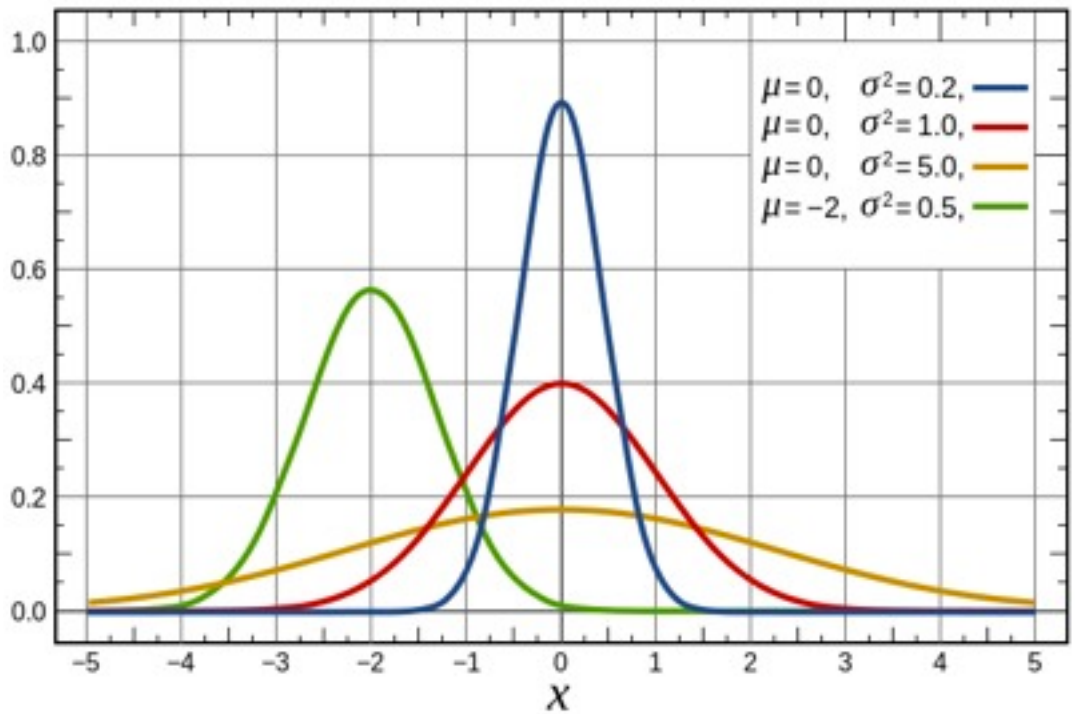
Effective Fisher matrix: Semi-Analytic approach to the Parameter Estimation Accuracy



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4th K-J workshop 2013. 06. 10 ~ 11

Outline

Fisher matrix (FM) for parameter estimation

: How to use the FM formalism for parameter estimation to calculate the measurement error.

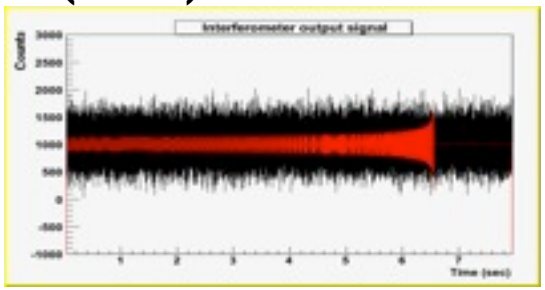
Effective Fisher matrix method

: What is the effective FM, why we need, how to calculate.
Comparison with the standard analytic FM.

Conclusion, on going work, future work

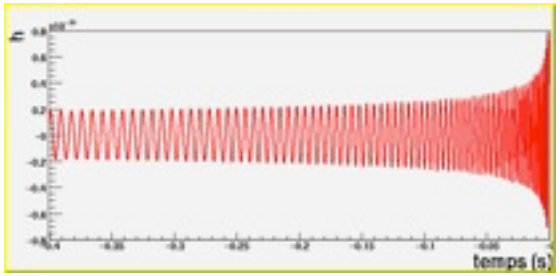
Parameter Estimation Analysis

Detector output (data)



$\otimes$

Theoretical waveform



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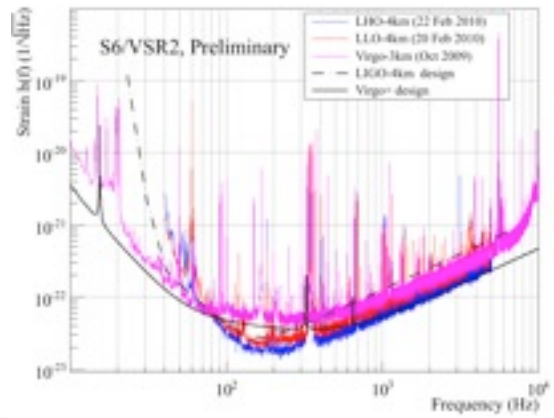
Detector noise sensitivity

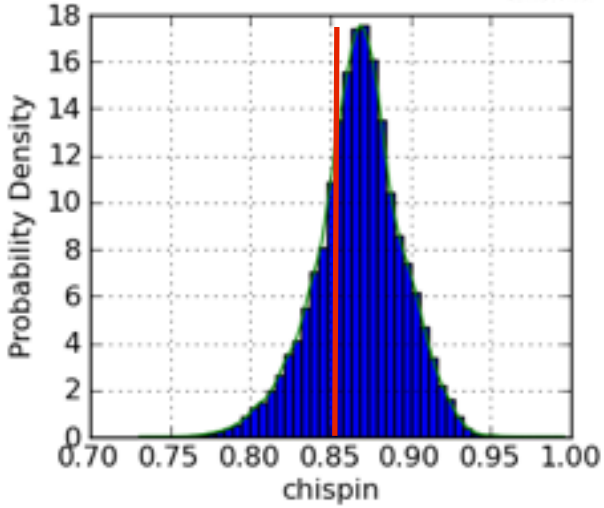


$$\tilde{d}(f) = \tilde{n}(f) + \tilde{s}(f)$$

9~15 parameters: mass, spin, position, ...

$$\langle d|h\rangle = 4\int_0^\infty \frac{df}{S_n(f)} \mathrm{Re}[\tilde{d}(f)\tilde{h}(f)^*]$$

==>

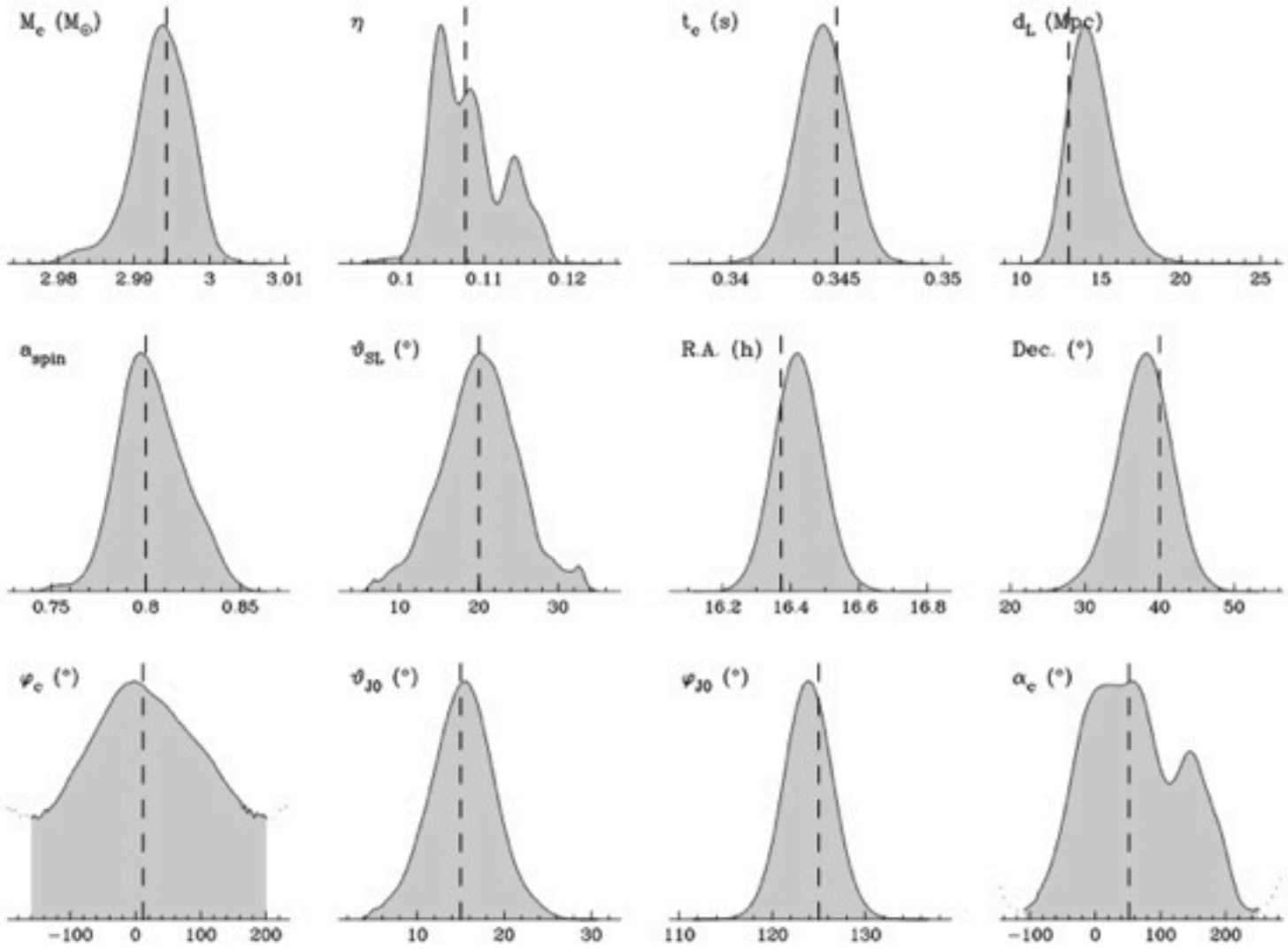


parameters

several million overlaps !!

==> Needs very large computational costs !!

Marginalized Posterior PDF (one spin binary)



Simple method? Fisher matrix

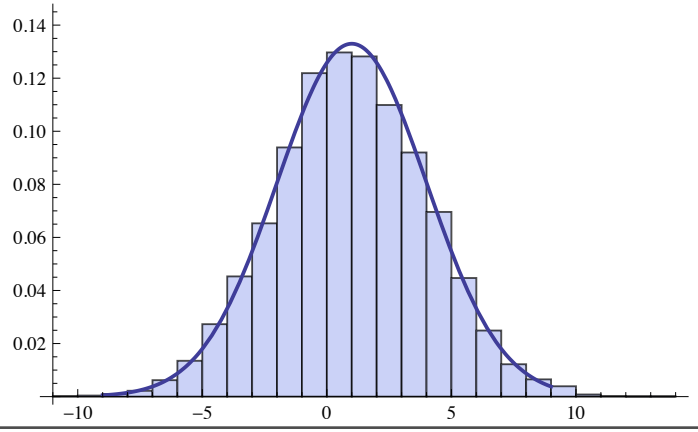
To find the real parameter values, we should do Parameter Estimation analysis. But very expensive and takes long time depending on binary models or waveform models (week ~ month).

Before PE analysis, can we estimate the measurement accuracies for a given signal?

Fisher matrix: analytic method, simpler, cheaper.

Where we assume Strong GW signal (High SNR) + Stationary, Gaussian noise

Then, the probability density function (PDF) of each parameter can be expressed by a Gaussian.



Properties of Gaussian function

1-D Gaussian $$p(\Delta x) \sim e^{-\Gamma \Delta x^2/2}$$

Variance $$\sigma^2 = \Gamma^{-1}$$

n-D multi-Gaussian $$p(\Delta \lambda_i) \sim e^{-\Gamma_{ij} \Delta \lambda_i \Delta \lambda_j/2}$$

Covariance matrix $$\Sigma_{ij} = \Gamma_{ij}^{-1}$$

Fisher Matrix (n by n)

Variance and correlation

$$\sigma^2 = \sqrt{\Sigma_{ii}} \qquad c_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$$

Standard Fisher matrix: fully analytic

$$h = h(\lambda_1, \lambda_2, \ldots.)$$

$$\Gamma_{ij} = \left\langle \frac{\partial h}{\partial \lambda_i} \middle| \frac{\partial h}{\partial \lambda_j} \right\rangle \qquad \langle d|h\rangle = 4\int_0^\infty \frac{df}{S_n(f)} \mathrm{Re}[\tilde{d}(f)\tilde{h}(f)^*]$$

Standard FM is based on differentiating a wave function w.r.t each parameter, so needs a analytic wave function.

But the post-Newtonian waveform is numerical data.

Post-Newtonian waveform: 3.5PN

$$h_+(t) = -\frac{2G\mu}{c^2 R}(1+\cos^2\iota)x(t)\cos 2\psi(t)$$

$$h_\times(t) = -\frac{2G\mu}{c^2 R}(2\cos\iota)x(t)\sin 2\psi(t)$$

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\eta(M\omega)^{5/3}\Bigg(1 - \frac{743+924\eta}{336}(M\omega)^{2/3} + \left(\frac{34\,103}{18\,144} + \frac{13\,661}{2016}\eta + \frac{59}{18}\eta^2\right)(M\omega)^{4/3} - \frac{1}{672}(4159+14\,532\eta)\pi(M\omega)^{5/3}\eta$$
$$+\Bigg[\left(\frac{16\,447\,322\,263}{139\,708\,800} - \frac{1712}{105}\gamma_E + \frac{16}{3}\pi^2\right) + \left(-\frac{273\,811\,877}{1\,088\,640} + \frac{451}{48}\pi^2 - \frac{88}{3}\hat{\theta}\right)\eta + \frac{541}{896}\eta^2 - \frac{5605}{2592}\eta^3$$
$$-\frac{856}{105}\log[16(M\omega)^{2/3}]\Bigg](M\omega)^2 + \left(-\frac{4\,415}{4\,032} + \frac{661\,775}{12\,096}\eta + \frac{149\,789}{3\,024}\eta^2\right)\pi(M\omega)^{7/3}\Bigg),$$

$$w(t) = \int_0^t \dot{w}(w)dt, \qquad \psi(t) = \int_0^t w(t)dt, \qquad x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$$

time domain data => FFT => frequency domain numerical data

SPA waveform (TaylorF2): approximation

$$h(f) \propto f^{-7/6} e^{i\psi(f)}$$ Stationary phase approximation

$$\begin{aligned}
\Phi_{\rm SPA}(f) &= 2\pi f t_c - \Phi_c - \frac{\pi}{4} + \frac{3}{128\nu v^5}\left\{1 + \frac{20}{9}\left(\frac{743}{336} + \frac{11}{4}\nu\right)v^2 - 16\pi v^3\right. \\
&+ 10\left(\frac{3058673}{1016064} + \frac{5429}{1008}\nu + \frac{617}{144}\nu^2\right)v^4 \qquad (3.20) \\
&+ \left(\frac{38645}{756} - \frac{65}{9}\nu\right)\left[1 + 3\log\left(\frac{v}{v_{\rm iso}}\right)\right]\pi v^5\left[\left(\frac{11583231236531}{4694215680}\right.\right. \\
&- \left.\frac{640}{3}\pi^2 - \frac{6848}{21}\gamma_E - \frac{6848}{21}\log(4v)\right) + \left(\frac{2255}{12}\pi^2 - \frac{15737765635}{3048192}\right)\nu \\
&+ \left.\left.\frac{76055}{1728}\nu^2 - \frac{127825}{1296}\nu^3\right]v^6 + \left(\frac{77096675}{254016} + \frac{378515}{1512}\nu - \frac{74045}{756}\nu^2\right)\pi v^7\right\}
\end{aligned}$$

frequency domain analytic function => Fisher matrix

$$\Gamma_{ij} = \left\langle \frac{\partial h}{\partial \lambda_i} \middle| \frac{\partial h}{\partial \lambda_j} \right\rangle$$

Why Effective FM?

SPA waveform is available for the inspiral motion,
so only valid for the low mass binaries.
High mass binary needs full IMR waveforms.

SPA waveform can only be used for the
non-precessing cases.

We need an Effective method to use the time domain
precessing waveforms or full IMR waveforms.

Effective Fisher matrix: semi-analytic

$$\Gamma_{ij} = \left\langle \frac{\partial h}{\partial \lambda_i} \middle| \frac{\partial h}{\partial \lambda_j} \right\rangle \equiv -\frac{\partial^2 \ln L(\lambda)}{\partial \lambda_i \partial \lambda_j}\bigg|_{\lambda_i = \lambda_j = \lambda_{\rm true}}$$

Derivative and Overlap:

Overlap and Derivative:

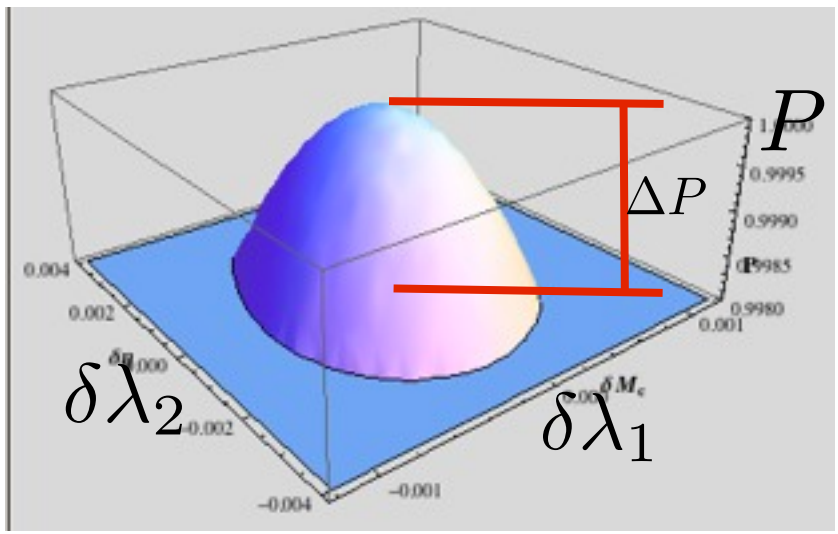
$$\ln L \propto -(1 - \langle h(\lambda) | h(\lambda + \delta\lambda) \rangle)$$

Overlap can be calculated numerically using both FD and TD waveforms.

How to calculate the Effective FM: 2-D case

Find the quadratic fitting function at a proper scale

$$P = 1 - [p_1(\delta\lambda_1)^2 + p_2(\delta\lambda_2)^2 + p_{12}(\delta\lambda_1)(\delta\lambda_2)]$$



$$\langle h(\lambda)|h(\lambda+\delta\lambda)\rangle$$

If $\Delta P \sim 0$,

Effective = Analytic

$$\Gamma_{ij} = -\frac{\partial^2 P}{\partial\lambda_i \partial\lambda_j} = \begin{vmatrix} 2p_1 & p_{12} \\ p_{12} & 2p_2 \end{vmatrix}$$

Binary model: no spin

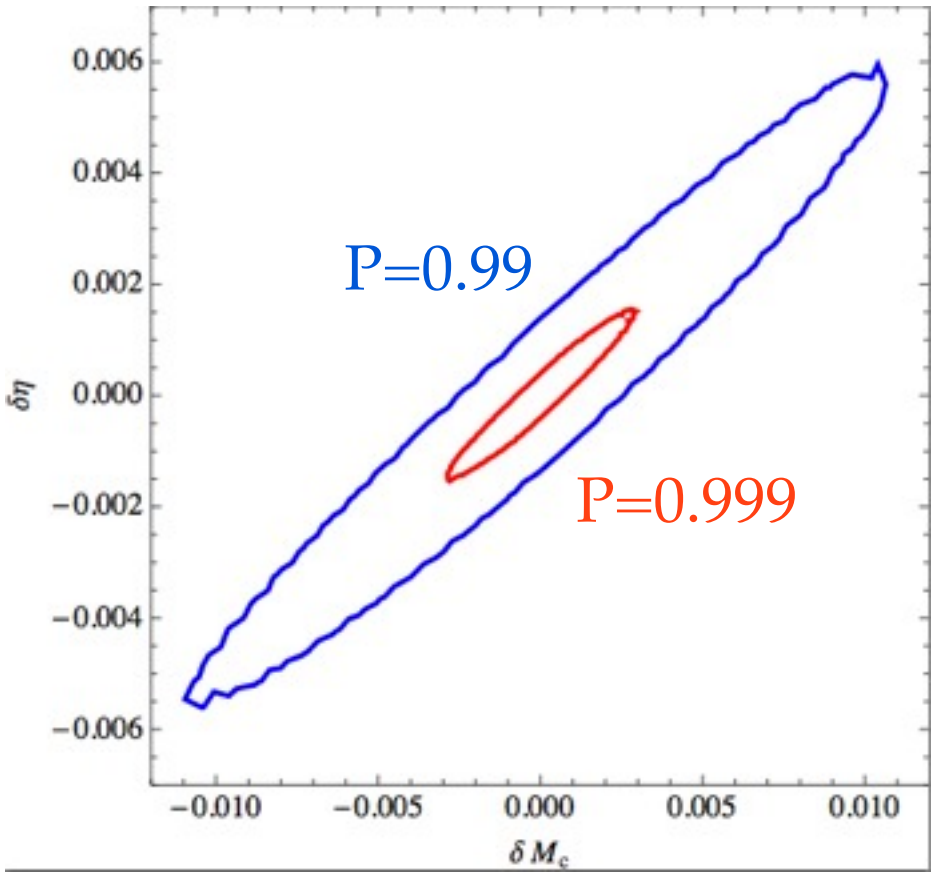
BH-NS binary model

i-LIGO nose curve

low frequency = 40Hz

SPA wave function

$$m_1 = 10\mathrm{M}_\odot, \quad m_2 = 1.4\mathrm{M}_\odot$$
$$\eta = m_1 m_2 / M^2 \sim 0.1$$
$$M_c = M\eta^{3/5} \sim 3$$



overlap contour is a clear ellipse: a quadratic function fits well

Comparison to Standard FM 1

Method		Effective						Standard	
Fitting scale		$\Delta P = 0.01$		$\Delta P = 0.005$		$\Delta P = 0.001$		$\Delta P \sim 0$	
Parameter		M_c	η	M_c	η	M_c	η	M_c	η
$(\hat{\Gamma}_{ij})_{\rm eff}$	M_c	4382.66	-7713.49	4461.44	-7827.53	4482.40	-7874.83	4491.01	-7884.31
	η	-	15042.1	-	15224.3	-	15334.1	-	15341.8
c_{ij}	M_c	1.00	0.9500	1.00	0.9498	1.00	0.9499	1.00	0.9498
	η	-	1.00	-	1.00	-	1.00	-	1.00
σ_i		0.00242	0.00131	0.00239	0.00130	0.00239	0.00129	0.00239	0.00129
$\lvert\Delta\sigma/\sigma\rvert$[%]		0.82	1.15	0.26	0.31	0.10	0.03	0.00	0.00

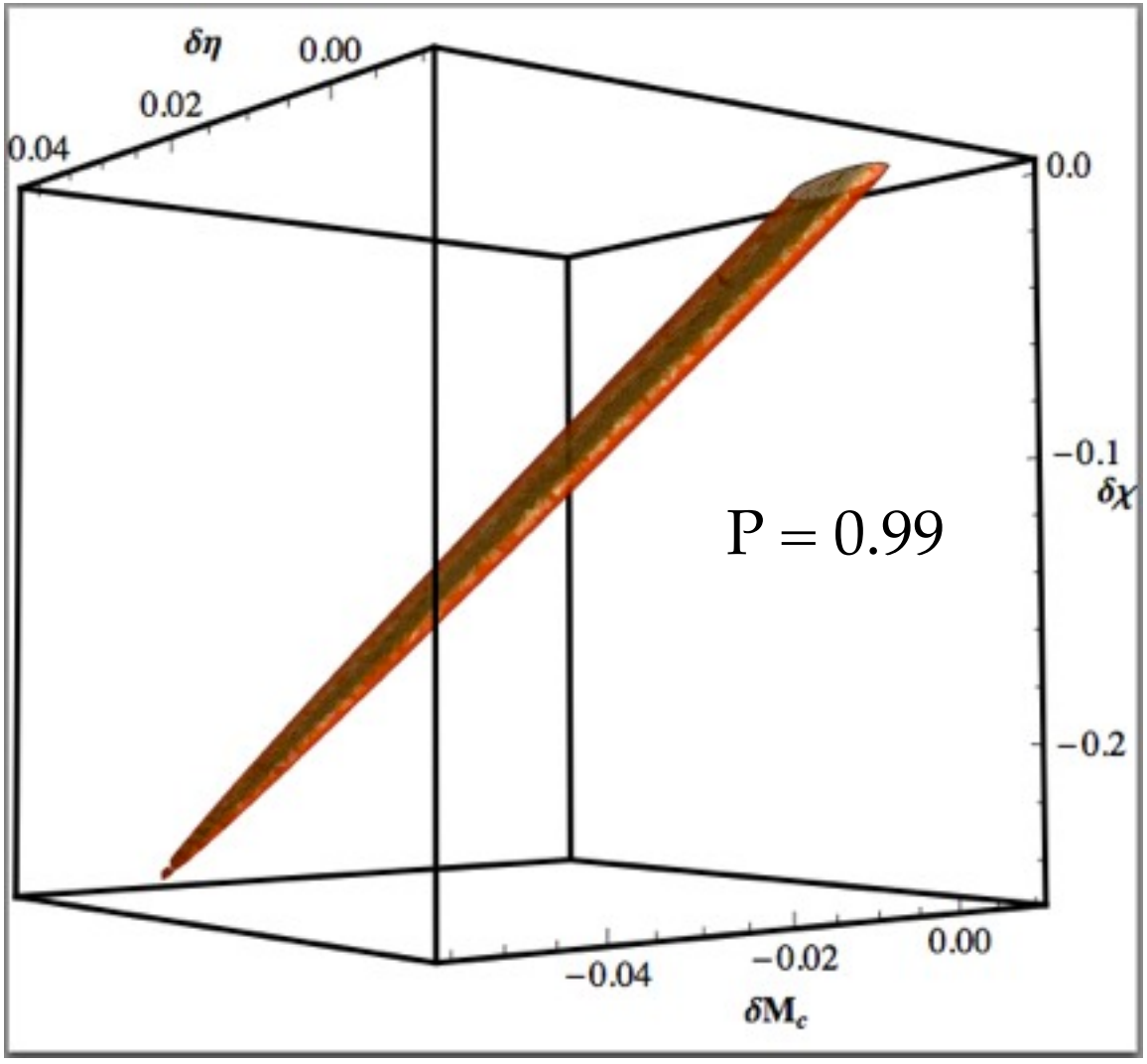
very good agreement at all scales !!

If $\Delta P \sim 0$,

Effective = Analytic

Binary model: aligned spin

BH-NS binary model, maximally rotating BH (a=1),
SPA wave function



long, thin ellipsoid

Comparison to Standard FM 2

Method		Effective									Standard		
Fitting scale		$\Delta P = 0.001$			$\Delta P = 0.0001$			$\Delta P = 0.00001$			$\Delta P \sim 0$		
Parameter		M_c	η	χ	M_c	η	χ	M_c	η	χ	M_c	η	χ
$(\hat{\Gamma}_{ij})_{\rm eff}$	M_c	4255.59	-492.231	-855.494	4929.62	-535.204	-1012.92	4953.975	-541.742	-1027.15	4937.52	-540.867	-1027.76
	η	-	201.220	138.073	-	214.770	150.570	-	219.357	153.338	-	220.952	153.784
	χ	-	-	182.849	-	-	218.990	-	-	223.836	-	-	224.798
c_{ij}	M_c	1.00	-0.9817	0.9985	1.00	-0.9779	0.9985	1.00	-0.9765	0.9985	1.00	-0.9761	0.9984
	η	-	1.00	-0.9878	-	1.00	-0.9843	-	1.00	-0.9833	-	1.00	-0.9829
	χ	-	-	1.00 -	-	-	1.00	-	-	1.00	-	-	1.00
$\sigma_i \times 10^2$		1.6528	2.6699	9.7267	1.5292	2.2667	8.6101	1.4958	2.1696	8.3284	1.4888	2.13775	8.24705
$\lvert\Delta\sigma/\sigma\rvert$[%]		11.01	24.94	17.94	2.71	6.07	4.40	0.47	1.53	0.99	0.00	0.00	0.00

good agreement only at the very fine scale!!

FD vs TD Effective FMs (no spin)

model		Effective FM				Conventional FM	
scale		$P > 0.99$		$P > 0.999$		$P \sim 1$	
parameter		M_c	η	M_c	η	M_c	η
$(\hat{\Gamma}_{ij})_{\rm eff}$	M_c	2899	-5295	2980	-5325	2546	-5313
	η	-	10261	-	10276	-	11954
c_{ij}	M_c	1.00	0.971	1.00	0.962	1.00	0.963
	η	-	1.00	-	1.00	-	1.00
σ_i		0.00776	0.00413	0.00673	0.00363	0.00735	0.00339

SpinTaylorT4 3.5pN waveform

similar results

Conclusions

Effective FM is consistent with the standard analytic FM for non spinning and aligned spin BH-NS binaries.

Using Effective FM, we can determine the PE accuracies for various time domain wave functions, e.g,. SpinTaylorT4, EOBNR, SEOBNR, etc...

On going work:

Comparison to the full PE analysis using time domain wave function.

To do:

We will investigate PE accuracies for the precessing BH-NS binaries using SpinTaylorT4 wave function and compare to the full MCMC results.

Collaboration item:

Using KAGRA noise curve ?