Quantum Properties of Idealized GW Detector

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Osaka University, June 10, 2013

The 4th Korea-Japan Workshop on KAGRA

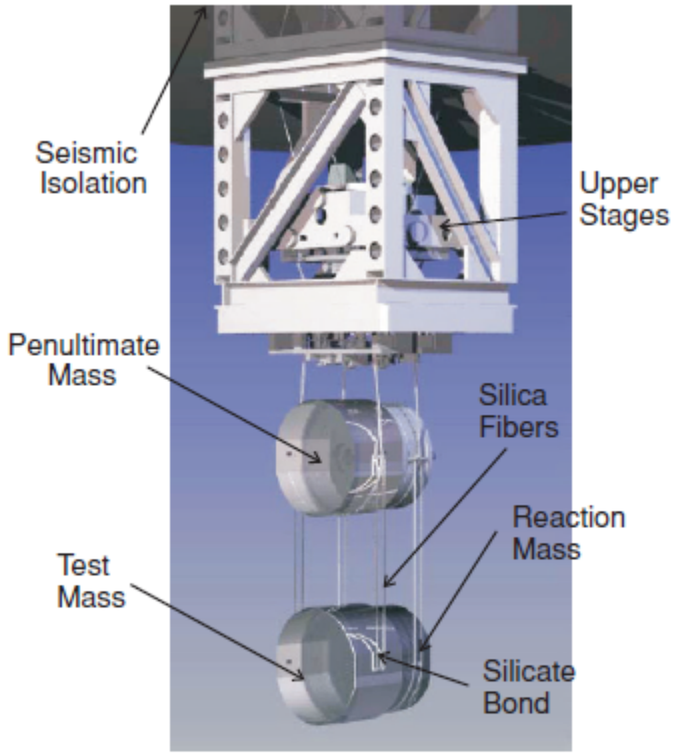
Outline

- Idealized Detector for Gravitational Waves
- Quantum Theory for Dissipation
- Winger Function of Time-Dependent Oscillator
- Damped Oscillator Driven by External Forces
- Conclusion

Idealized Detector for GW

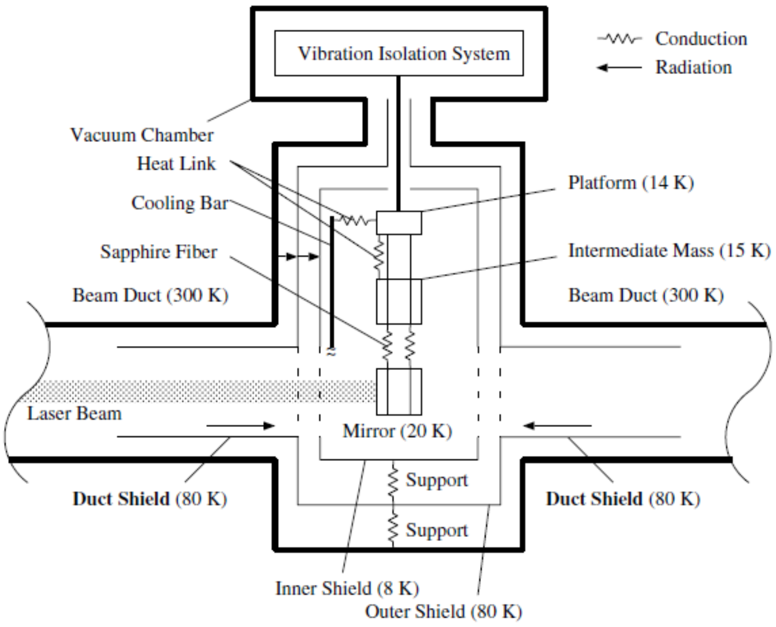
Schematic of GW Detectors

Advanced LIGO



[Harry et al, CQG 27 ('10)]♪

KAGRA



[Sakakibara et al, CQG 29 ('12)]♪

Idealized Detector

- A detector mass is much smaller than a wavelength of GW traveling along the z-direction [MTW, Gravitation ('73)]

$$\left(\frac{d^2x}{dt^2}\right)_{\text{due to GW}} = -R_{x0j0}x^j = -\frac{1}{2}\omega^2 a_+ e^{-i\omega t} x$$

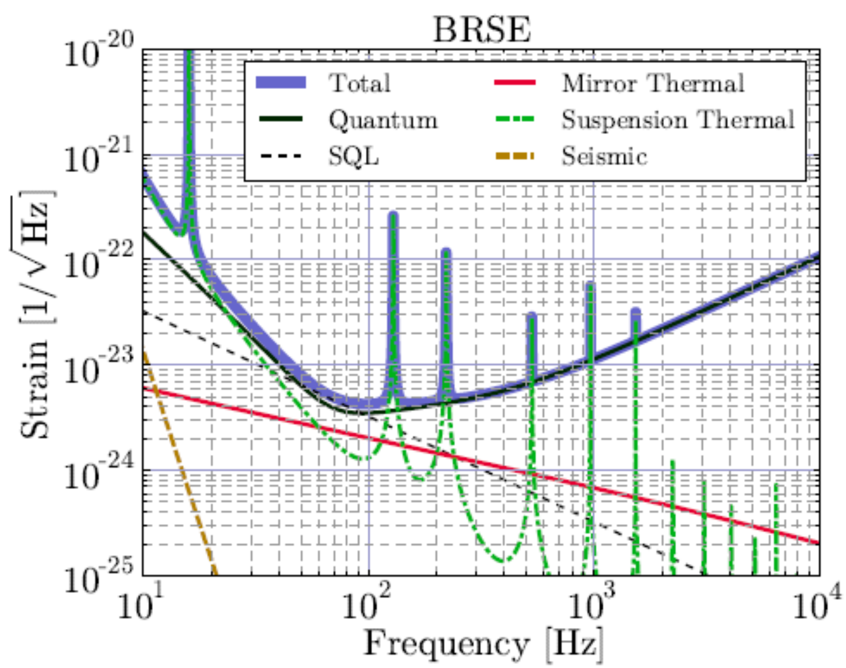
$$\left(\frac{d^2y}{dt^2}\right)_{\text{due to GW}} = -R_{y0j0}x^j = +\frac{1}{2}\omega^2 a_+ e^{-i\omega t} y$$

- An ideal detector of two masses connected by a spring and separated by ξ along the x-direction is a damped oscillator driven by GW

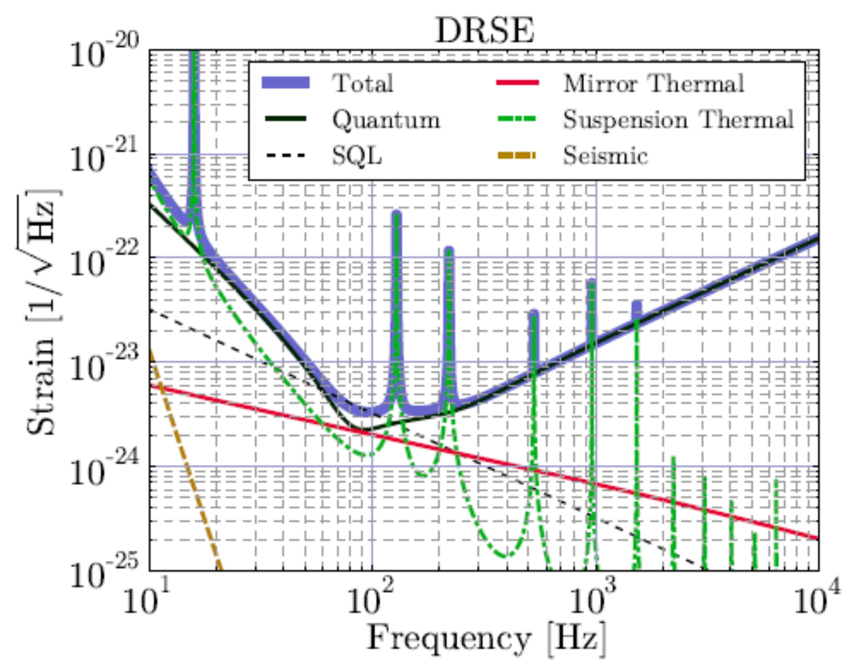
$$\ddot{\xi} + \gamma\dot{\xi} + \omega_0^2\xi = -\frac{1}{2}\omega^2 a_+ L e^{-i\omega t} \underbrace{\sin^2\theta\cos 2\phi}_{\text{rotation angle}}$$

Estimated Noises of KAGRA

Broad-band Resonant Side band Extraction



Detuned Resonant Sideban d Extraction



[Aso, Michimura, Somyia, "Interferometer design of KAGRA …" ('13)]♪

Why Quantum Theory for Detector ?

- The intensity of GW reaches the standard quantum limit (SQL).
  - Can one beat the standard quantum limit?

- Quantum nondemolition (QND) continuous measurements matter
  - What quantum states and how the measurements?

Quantum Theory for Dissipation

CK Oscillator

- Caldirola ('41) and Kanai ('48) introduced an osci llator with a time-dependent mass

$$H_{CK}(t) = \frac{1}{2me^{\gamma t}} p^2 + \frac{me^{\gamma t}\omega_0^2}{2} x^2$$

- Classical motion is a damped oscillator

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$$

- Quantum theory is the Schrodinger equation [Sc hrodinger picture in this talk]

$$i\hbar \frac{\partial}{\partial t} \Psi(t) = \hat{H}_{CK}(t) \Psi(t)$$

Construction of Quantum States

- The time-dependent annihilation and creation operators, quantum invariant operators [Malkin et al ('70); SPK, Lee, PRD ('00); SPK, Page, PRA ('01)] and may be of use for QND, $a\uparrow + (t)a\,(t)$ for photon number or amplitude

$$\hat{a}_u(t) = \frac{i}{\sqrt{\hbar}}\left[u^*(t)\hat{p} - me^{\gamma t}\dot{u}^*(t)\hat{x}\right], \quad \hat{a}_u^+(t) = \mathrm{H.C.}$$

$$i\hbar\frac{d\hat{a}_u(t)}{dt} = i\hbar\frac{\partial \hat{a}_u(t)}{\partial t} + \left[\hat{a}_u(t), \hat{H}_{CK}(t)\right] = 0$$

$$\ddot{u} + \gamma\dot{u} + \omega_0^2 u = 0, \quad me^{\gamma t}\left(u\dot{u}^* - \dot{u}u^*\right) = i$$

- The general time-dependent quantum states

$$\hat{a}_u(t)\Psi_0(x,t) = 0, \quad \Psi_n(x,t) = \frac{\left(\hat{a}_u^+(t)\hat{a}_u(t)\right)^n}{\sqrt{n!}}\Psi_0(x,t)$$

Pseudo-Stationary States

- The pseudo-stationary states from $u_0 = e^{-\gamma t/2} \times e^{-i\Omega t}/\sqrt{2\hbar m\Omega}$ [Srivastava et al ('91); SPK, J PA ('03)]

$$\Psi_n(x,t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\Omega e^{\gamma t}}{\pi\hbar} \right)^{1/4} e^{-i\Omega(n+1/2)t} H_n\left(\sqrt{\frac{m\Omega e^{\gamma t}}{\hbar}} x \right)$$

$$\times \exp\left[-\frac{m\Omega e^{\gamma t}}{2\hbar} \left(1 + i\frac{\gamma}{2\Omega} \right) x^2 \right], \quad \Omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

- The dispersion relations of the ground state (SQL)

$$\langle \hat{x}^2 \rangle = \frac{\hbar}{2m\Omega} e^{-\gamma t}, \quad \langle \hat{p}^2 \rangle = \frac{\hbar m\omega_0^2}{2\Omega} e^{\gamma t}$$

Density Matrix, Quantum Decoherence and Correlation

- The density matrix [SPK et al ('03)]

$$\rho_{CK}(x',x,t) = \Psi_0(x,t)\Psi_0^*(x',t)$$
$$= \left(m\Omega e^{\gamma t}/\pi\hbar\right)^{1/2} \exp\left[-\Gamma_c x_c^2 - \Gamma_\delta x_\delta^2 - \Gamma_\mu x_c x_\delta\right]$$
$$x_c = \left(x'+x\right)/2, \quad x_\delta = \left(x'-x\right)/2$$

- The measure of quantum decoherence [Morikawa ('90)] and classical correlation [Guilini et al ('96)]

$$\delta_{QD} = \left(1/2\right)\sqrt{\Gamma_c/\Gamma_\delta} = 1/2$$
$$\delta_{CC} = \left(1/2\right)\sqrt{\Gamma_c^2\Gamma_\delta^2/\Gamma_\mu^*\Gamma_\mu} = \left(m^2\Omega^2/\hbar\gamma\right)e^{\gamma t}$$

Density Matrix, Quantum Decoherence and Correlation

- Quantum coherence (no decoherence)

$$\delta_{QD} = (1/2)\sqrt{\Gamma_c / \Gamma_\delta} << 1$$

- Classically NOT correlated

$$\delta_{CC} = (1/2)\sqrt{\Gamma_c^2 \Gamma_\delta^2 / \Gamma_\mu^* \Gamma_\mu} >> 1$$

- CK oscillator shows no quantum decoherence and is classically correlated when $t \ll t_*$ but classically NOT correlated when $t \gg t_*$

$$t_* = \frac{m}{\gamma} \ln\left(\frac{\hbar\gamma}{(m\Omega)^2} \right)$$

Squeezed Pseudo-Stationary States

- The general time-dependent states [SPK, Lee PR D ('00); SPK, Page, PRA ('01)]

$$\Psi_n(x,t) = \frac{1}{\sqrt{2^n n! \sqrt{2\pi\hbar u^* u}}} \left(\frac{u}{\sqrt{u^* u}} \right)^{n+1/2} H_n\left(\frac{x}{\sqrt{2\hbar u^* u}} \right) e^{i \frac{m e^{\gamma t} \dot{u}^*}{2\hbar u^*} x^2}$$

- The squeezed states and dispersion relations

$$u_{r\phi}(t) = \left(\cosh r\right) u_0(t) + \left(\sinh r\, e^{i\phi}\right) u_0^*(t), \quad u_0(t) = \frac{e^{-\gamma t/2}}{\sqrt{2\hbar m \Omega}} e^{-i\Omega t}$$

$$\left\langle \Psi_n \left| \hat{x}^2 \right| \Psi_n \right\rangle = \hbar u_{r\phi}^* u_{r\phi} (2n+1), \ \left\langle \Psi_n \left| \hat{p}^2 \right| \Psi_n \right\rangle = \hbar m^2 e^{2\gamma t} \dot{u}_{r\phi}^* \dot{u}_{r\phi} (2n+1)$$

BFT Oscillator

- Bateman ('31) and Feshbach-Tikochinsky ('77) also introduced a closed system for a damped oscillator

$$H_{BFT} = \frac{1}{m} p_x p_y + \frac{\gamma}{2}\left(y p_y - x p_x\right) + m\Omega^2 xy \,, \left(\Omega^2 = \omega_0^2 - \frac{\gamma^2}{4}\right)$$

- The classical motions

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \quad \text{(damped oscillator)}$$

$$\ddot{y} - \gamma\dot{y} + \omega_0^2 y = 0 \quad \text{(amplified oscillator)}$$

Reduced Density Matrix

- The reduced density matrix for the damped oscilla tor [SPK et al, JKPS ('03)]

$$\rho(x',x) = N\exp\left[-e^{\gamma t}\left\{\left(A^* x'^2 + A x^2\right) - \frac{\left(Dx' + D^* x\right)^2}{4\left(A^* + A\right)}\right\}\right]$$

$$A(\gamma) = \sqrt{\left(\frac{m\Omega}{2\hbar}\right)^2 + \frac{\left|D(\gamma)\right|^2}{4}}\, e^{i(\pi-\theta)}, \quad D(\gamma) = \left|D(\gamma)\right| e^{i\theta}$$

$$N = \sqrt{\frac{e^{\gamma t}}{\pi\left(A^* + A\right)}}\left[\left(A^* + A\right)^2 + \frac{1}{4}\left(D^* + D\right)^2\right]$$

Quantum Decoherence and Classical Correlation

- The measure quantum decoherence and classical correlation

$$\delta_{QD} = \frac{1}{2}\sqrt{\frac{(m\Omega/\hbar)^2}{(m\Omega/\hbar)^2\cos^2\theta + |D|^2}}$$

$$\delta_{CC} = \frac{1}{2}|\cot\theta|\left[(m\Omega/\hbar)^2\cos^2\theta + |D|^2\right]$$

- In the case of large dissipation ($\theta \approx \pi/2$ and $|D(\gamma)| \gg m\Omega/\hbar$), a significant degree of quantum decoherence $\delta_{QD} \ll 1/2$ and classical correlation $\delta_{CC} \ll 1$, and thus a loss of unitarity.

Wigner Function of Time-Dependent Oscillator

Time-Dependent Generalized Oscillat or

- Time-dependent generalized oscillator

$$\hat{H}(t) = \frac{X(t)}{2}\hat{p}^2 + \frac{Y(t)}{2}\left(\hat{p}\hat{x} + \hat{x}\hat{p}\right) + \frac{Z(t)}{2}\hat{x}^2$$

- Classical motion is a damped oscillator

$$\frac{d}{dt}\left(\frac{\dot{u}}{X}\right) + \left[XZ - Y^2 + \frac{\dot{X}Y - X\dot{Y}}{X}\right]\left(\frac{u}{X}\right) = 0$$

- Time-dependent annihilation/ creation operators, i nvariant operators [SPK, Page, PRA ('01)]

$$\hat{a}(t) = \frac{i}{\sqrt{\hbar}}\left[u^*(t)\hat{p} - \frac{1}{X(t)}\left\{\dot{u}^*(t) - Y(t)u^*(t)\right\}\hat{x}\right], \quad \hat{a}^+(t) = \text{H.C.}$$

Density Operator

- The most general density operator [SPK, Page, P LB ('13)]

$$\hat{\rho}(t) = \frac{1}{Z}\exp\left[-\beta\left(\hbar\omega_0\hat{a}^+(t)\hat{a}(t) + \delta\hat{a}^+(t) + \delta^*\hat{a}(t)\right)\right]$$

- The density operator has five parameters:
  - a real constant $\beta\omega_0$ for an initial thermal distribution
  - a complex constant δ for the classical position and mo mentum
  - one squeeze parameter and one squeeze angle for u

Dispersion Relations

- The dispersion relation w.r.t. the density operator

$$\left\langle \hat{O} \right\rangle = \mathrm{Tr}\left[\hat{O}\hat{\rho}(t)\right]$$

- The position and momentum expectation values

$$\left\langle \hat{x} \right\rangle = x_c(t), \qquad \left\langle \hat{p} \right\rangle = p_c(t)$$

- The dispersion relations

$$\left\langle \hat{x}^2 \right\rangle - x_c^2(t) = \hbar u^* u\left(2\bar{n}+1\right), \qquad \bar{n} = \frac{1}{e^{\beta\hbar\omega_0}-1}$$

$$\left\langle \hat{p}^2 \right\rangle - p_c^2(t) = \frac{\hbar}{X}\left(\dot{u}^* - Yu^*\right)\left(\dot{u} - Yu\right)\left(2\bar{n}+1\right)$$

Density Matrix and Wigner Function

- The density matrix $\rho(x,x\uparrow') = x\rho(t)x'$
- The Winger function

$$P(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy\,\rho(x-y,x+y)e^{2ipy/\hbar} = P_T(x-x_c,p-p_c)$$

- The thermal Wigner function

$$P_T(x,p) = \frac{\tanh(\beta\hbar\omega_0/2)}{\pi\hbar}\exp\left[-\frac{\tanh(\beta\hbar\omega_0/2)}{\hbar\omega_0}H_E(x,p)\right]$$

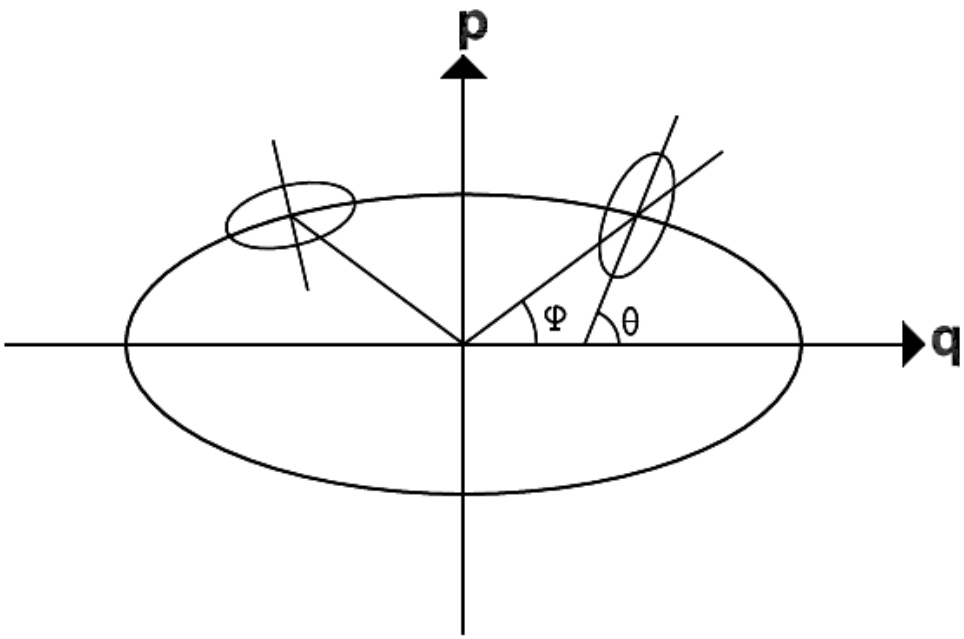
$$H_E(x,p) = \omega_0|u|^2\left(p - \frac{d\ln|u|}{dt}\frac{x}{X}\right)^2 + \frac{\omega_0}{4|u|^2}x^2$$

Wigner Function

- The general solution to a harmonic oscillator

$$u_{r\phi}(t) = \frac{1}{\sqrt{2m\omega_0}}\left[(\cosh r)e^{-i\omega_0 t} + (\sinh r e^{i\phi})e^{i\omega_0 t}\right]$$

- The contour of the Winger function



Damped Oscillator Driven by External Forces

Driven Damped Oscillator

- CK damped oscillator driven by external forces

$$H(t) = H_{CK}(t) - F(t)x$$

- The classical motion is a damped oscillator

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F(t)$$

- The external forces include gravitational waves and (thermal) noises.
- The quantum theory is still the time-dependent Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(t) = \hat{H}(t)\Psi(t)$$

Construction of Quantum States

- Two linear invariant operators [SPK, CMP ('03)]

$$\hat{b}(t) = \hat{a}_u(t) - \frac{i}{\sqrt{\hbar}} f(t)\hat{e}, \quad \hat{b}^+(t) = \mathrm{H.C.}$$

$$\ddot{u} + \gamma\dot{u} + \omega_0^2 u = 0, \quad me^{\gamma t}\left(u\dot{u}^* - \dot{u}u^*\right) = i$$

$$f(t) = \int_{-\infty}^{t} F(t')u^*(t')dt'$$

- The ground state is a coherent state of CK oscillator and has the same dispersions (minimal uncertainty)

$$\hat{b}(t)\Psi_0(x,t) = 0 \quad \Leftrightarrow \hat{a}_u(t)\Psi_0(x,t) = \frac{i}{\sqrt{\hbar}} f(t)\Psi_0(x,t)$$

Displaced Number States

- The displacement operator [SPK, JKPS ('04)]

$$\hat{b}_u(t)|\alpha,0,t\rangle = 0 \quad \Leftrightarrow \hat{a}_u(t)|\alpha,0,t\rangle = \alpha|\alpha,0,t\rangle \quad \left(\alpha = \frac{i}{\sqrt{\hbar}} f(t)\right)$$

$$|\alpha,0,t\rangle = \hat{D}_u(\alpha)|0,t\rangle, \qquad \hat{D}_u(\alpha) = \exp\left[\alpha\hat{a}_u(t) - \alpha^*\hat{a}_u^+(t)\right]$$

- The displaced number states (MUS)

$$|n,t\rangle = \frac{\left(\hat{a}_u^+\hat{a}_u\right)^n}{\sqrt{n!}}|0,t\rangle \Rightarrow |\alpha,n,t\rangle = \hat{D}_u(\alpha)|n,t\rangle$$

Centroid and Dispersions

- The centroid

$$x_c(t) = \langle \alpha, n, t | \hat{x} | \alpha, n, t \rangle = \sqrt{\hbar}\left[\alpha u + \alpha^* u^*\right]$$

$$p_c(t) = \langle \alpha, n, t | \hat{p} | \alpha, n, t \rangle = \sqrt{\hbar} m e^{\gamma t}\left[\alpha \dot{u} + \alpha^* \dot{u}^*\right]$$

- The dispersion relations are the same as detector

$$\langle \alpha, n, t | \hat{x}^2 | \alpha, n, t \rangle - x_c^2(t) = \hbar u^* u (2n+1)$$

$$\langle \alpha, n, t | \hat{p}^2 | \alpha, n, t \rangle - p_c^2(t) == \hbar m^2 e^{2\gamma t} \dot{u}^* \dot{u} (2n+1)$$

Conclusion

- Study the quantum states of a damped oscillator as an ideal detector for gravitational waves
  - Minimal uncertainty states (MUS)
  - Squeezed states: $(\Delta x \Delta p)\downarrow SQ \geq (\Delta x \Delta p)\downarrow MUS$ but may b eat SQL
  - Quantum invariant operators for QND measurements
- Discuss two quantum dissipation models for a dam ped oscillator: CK oscillator vs BFT oscillator
  - Quantum decoherence
  - Classical correlation
- Provide the Wigner function for the phase-space for mulation of an ideal detector