

Quantum Properties of Gravitational Wave Detector

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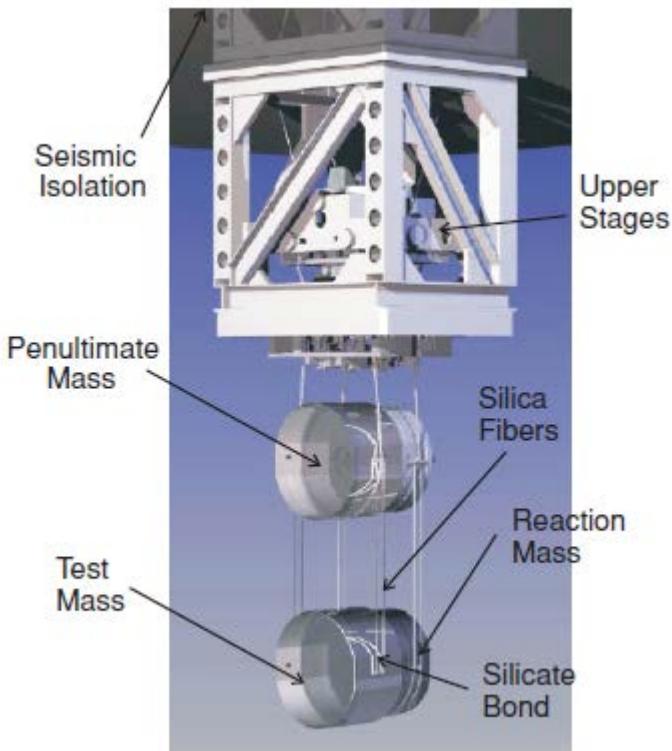
Outline

- Idealized Detector for Gravitational Waves
- Quantum Theory for Dissipation
- Winger Function of Time-Dependent Oscillator
- Damped Oscillator Driven by External Forces
- Conclusion

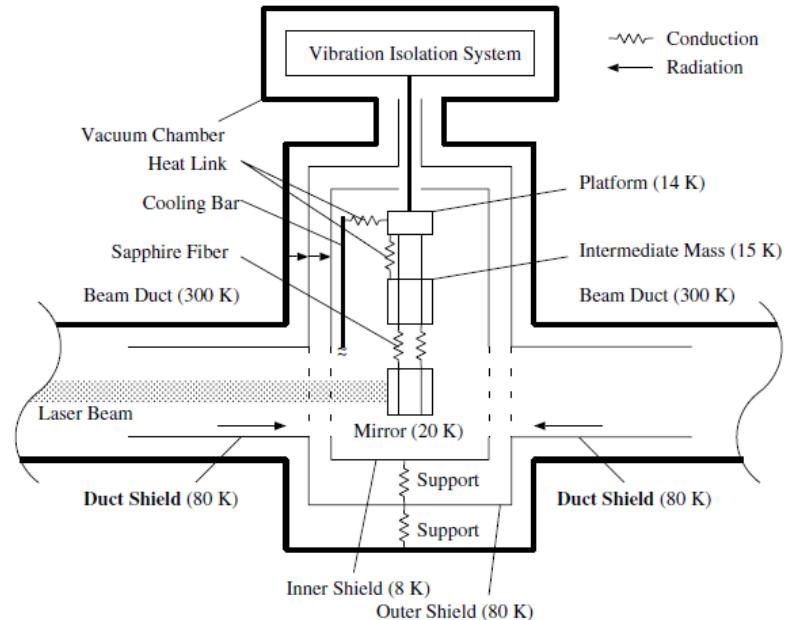
Idealized Detector for GW

Schematic of GW Detectors

Advanced LIGO



KAGRA



[Harry et al, CQG 27 ('10)]

[Sakakibara et al, CQG 29 ('12)]

Idealized Detector

- A detector mass is much smaller than a wavelength of GW traveling along the z-direction [MTW, Gravitation ('73)]

$$\left(\frac{d^2 x}{dt^2} \right)_{\text{due to GW}} = -R_{x0j0} x^j = -\frac{1}{2} \omega^2 a_+ e^{-i\omega t} x$$

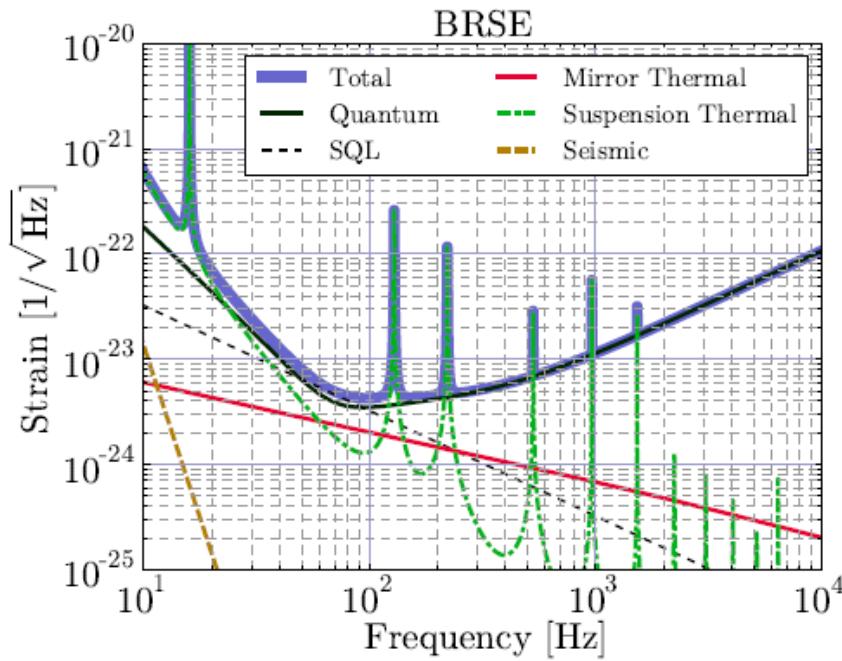
$$\left(\frac{d^2 y}{dt^2} \right)_{\text{due to GW}} = -R_{y0j0} x^j = +\frac{1}{2} \omega^2 a_+ e^{-i\omega t} y$$

- An ideal detector of two masses connected by a spring and separated by ξ along the x-direction is **a damped oscillator driven by GW**

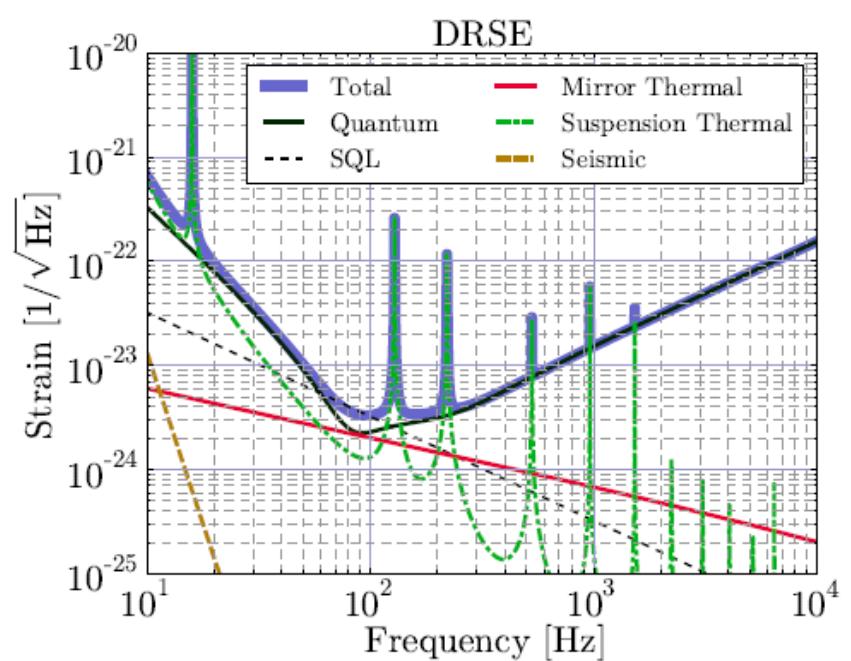
$$\ddot{\xi} + \gamma \dot{\xi} + \omega_0^2 \xi = -\frac{1}{2} \omega^2 a_+ L e^{-i\omega t} \underbrace{\sin^2 \theta \cos 2\phi}_{\text{rotation angle}}$$

Estimated Noises of KAGRA

Broad-band Resonant Sideband Extraction



Detuned Resonant Sideband Extraction



[Aso, Michimura, Somyia, “Interferometer design of KAGRA ...” ('13)]

Why Quantum Theory for Detector?

- The intensity of GW reaches the standard quantum limit (SQL).
 - Can one beat the standard quantum limit?
- Quantum nondemolition (QND) continuous measurements matter
 - What quantum states and how the measurements?

Quantum Theory for Dissipation

CK Oscillator

- Caldriola ('41) and Kanai ('48) introduced an oscillator with a time-dependent mass

$$H_{CK}(t) = \frac{1}{2me^{\pi t}} p^2 + \frac{me^{\pi t}\omega_0^2}{2} x^2$$

- The classical motion is a damped oscillator
- The quantum theory is the time-dependent Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(t) = \hat{H}_{CK}(t) \Psi(t)$$

Construction of Quantum States

- The time-dependent annihilation and creation operators, **quantum invariant operators** [Malkin et al ('70); SPK, Lee, PRD ('00); SPK, Page, PRA ('01)]

$$\hat{a}_u(t) = \frac{i}{\sqrt{\hbar}} [u^*(t) \hat{p} - m e^\gamma \dot{u}^*(t) \hat{x}], \quad \hat{a}_u^+(t) = \text{H.C.}$$

$$i\hbar \frac{d\hat{a}_u(t)}{dt} = i\hbar \frac{\partial \hat{a}_u(t)}{\partial t} + [\hat{a}_u(t), \hat{H}_{CK}(t)] = 0$$

$$\ddot{u} + \gamma \dot{u} + \omega_0^2 u = 0, \quad m e^\gamma (u \dot{u}^* - \dot{u} u^*) = i$$

- The general time-dependent quantum states

$$\hat{a}_u(t) \Psi_0(x, t) = 0, \quad \Psi_n(x, t) = \frac{(\hat{a}_u^+(t) \hat{a}_u(t))^n}{\sqrt{n!}} \Psi_0(x, t)$$

Pseudo-Stationary States

- The pseudo-stationary states from $u_0 = e^{-\gamma t/2} \times e^{-i\Omega t}/\sqrt{2\hbar m\Omega}$ [Srivastava et al ('91); SPK, JPA ('03)]

$$\Psi_n(x, t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\Omega e^{\gamma t}}{\pi\hbar} \right)^{1/4} e^{-i\Omega(n+1/2)t} H_n \left(\sqrt{\frac{m\Omega e^{\gamma t}}{\hbar}} x \right)$$
$$\times \exp \left[-\frac{m\Omega e^{\gamma t}}{2\hbar} \left(1 + i \frac{\gamma}{2\Omega} \right) x^2 \right], \quad \Omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

- The dispersion relations of the ground state (SQL)

$$\langle \hat{x}^2 \rangle = \frac{\hbar}{2m\Omega} e^{-\gamma t}, \quad \langle \hat{p}^2 \rangle = \frac{\hbar m \omega_0^2}{2\Omega} e^{\gamma t}$$

Density Matrix, Quantum Decoherence and Correlation

- The density matrix [SPK et al ('03)]

$$\begin{aligned}\rho_{CK}(x', x, t) &= \Psi_0(x, t)\Psi_0^*(x', t) \\ &= \left(m\Omega e^\pi / \pi\hbar\right)^{1/2} \exp\left[-\Gamma_c x_c^2 - \Gamma_\delta x_\delta^2 - \Gamma_\mu x_c x_\delta\right] \\ x_c &= (x' + x)/2, \quad x_\delta = (x' - x)/2\end{aligned}$$

- The measure of quantum decoherence [Morikawa ('90)] and classical correlation [Guilini et al ('96)]

$$\begin{aligned}\delta_{QD} &= (1/2)\sqrt{\Gamma_c/\Gamma_\delta} = 1/2 \\ \delta_{CC} &= (1/2)\sqrt{\Gamma_c^2\Gamma_\delta^2/\Gamma_\mu^*\Gamma_\mu} = \left(m^2\Omega^2/\hbar\gamma\right)e^\pi\end{aligned}$$

Density Matrix, Quantum Decoherence and Correlation

- Quantum coherence (no decoherence)

$$\delta_{QD} = (1/2)\sqrt{\Gamma_c / \Gamma_\delta} \ll 1$$

- Classically NOT correlated

$$\delta_{CC} = (1/2)\sqrt{\Gamma_c^2 \Gamma_\delta^2 / \Gamma_\mu^* \Gamma_\mu} \gg 1$$

- CK oscillator shows no quantum decoherence and is classically correlated when $t \ll t_*$ but classically NOT correlated when $t \gg t_*$

$$t_* = \frac{m}{\gamma} \ln \left(\frac{\hbar\gamma}{(m\Omega)^2} \right)$$

Squeezed Pseudo-Stationary States

- The general time-dependent states [SPK, Lee PRD ('00); SPK, Page, PRA ('01)]

$$\Psi_n(x, t) = \frac{1}{\sqrt{2^n n! \sqrt{2\pi\hbar u^* u}}} \left(\frac{u}{\sqrt{u^* u}} \right)^{n+1/2} H_n \left(\frac{x}{\sqrt{2\hbar u^* u}} \right) e^{i \frac{me^\gamma \dot{u}^*}{2\hbar u^*} x^2}$$

- The squeezed states and dispersion relations

$$u_{r\phi}(t) = (\cosh r)u_0(t) + (\sinh r e^{i\phi})u_0^*(t), \quad u_0(t) = \frac{e^{-\gamma t/2}}{\sqrt{2\hbar m\Omega}} e^{-i\Omega t}$$

$$\langle \Psi_n | \hat{x}^2 | \Psi_n \rangle = \hbar u_{r\phi}^* u_{r\phi} (2n+1), \quad \langle \Psi_n | \hat{p}^2 | \Psi_n \rangle = \hbar m^2 e^{2\gamma t} \dot{u}_{r\phi}^* \dot{u}_{r\phi} (2n+1)$$

BFT Oscillator

- Bateman ('31) and Feshbach-Tikochinsky ('77) also introduced a closed system for a damped oscillator

$$H_{BFT} = \frac{1}{m} p_x p_y + \frac{\gamma}{2} (y p_y - x p_x) + m \Omega^2 xy, \left(\Omega^2 = \omega_0^2 - \frac{\gamma^2}{4} \right)$$

- The classical motions

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0 \quad (\text{damped oscillator})$$

$$\ddot{y} - \gamma \dot{y} + \omega_0^2 y = 0 \quad (\text{amplified oscillator})$$

Reduced Density Matrix

- The reduced density matrix for the damped oscillator [SPK et al, JKPS ('03)]

$$\rho(x', x) = N \exp \left[-e^{\eta t} \left\{ (A^* x'^2 + A x^2) - \frac{(Dx' + D^* x)^2}{4(A^* + A)} \right\} \right]$$

$$A(\gamma) = \sqrt{\left(\frac{m\Omega}{2\hbar} \right)^2 + \frac{|D(\gamma)|^2}{4}} e^{i(\pi-\theta)}, \quad D(\gamma) = |D(\gamma)| e^{i\theta}$$

$$N = \sqrt{\frac{e^{\eta t}}{\pi(A^* + A)}} \left[(A^* + A)^2 + \frac{1}{4}(D^* + D)^2 \right]$$

Quantum Decoherence and Classical Correlation

- The measure quantum decoherence and classical correlation

$$\delta_{QD} = \frac{1}{2} \sqrt{\frac{(m\Omega/\hbar)^2}{(m\Omega/\hbar)^2 \cos^2 \theta + |D|^2}}$$

$$\delta_{CC} = \frac{1}{2} |\cot \theta| \left[(m\Omega/\hbar)^2 \cos^2 \theta + |D|^2 \right]$$

- In the case of large dissipation ($\theta \approx \frac{\pi}{2}$ and $|D(\gamma)| \gg m\Omega/\hbar$), a significant degree of quantum decoherence $\delta_{QD} \ll 1/2$ and classical correlation $\delta_{CC} \ll 1$, and thus a loss of unitarity.

Wigner Function of Time-Dependent Oscillator

Time-Dependent Generalized Oscillator

- Time-dependent generalized oscillator

$$\hat{H}(t) = \frac{X(t)}{2} \hat{p}^2 + \frac{Y(t)}{2} (\hat{p}\hat{x} + \hat{x}\hat{p}) + \frac{Z(t)}{2} \hat{x}^2$$

- Classical motion is a damped oscillator

$$\frac{d}{dt} \left(\frac{\dot{u}}{X} \right) + \left[XZ - Y^2 + \frac{\dot{X}Y - X\dot{Y}}{X} \right] \left(\frac{u}{X} \right) = 0$$

- Time-dependent annihilation/ creation operators, invariant operators [SPK, Page, PRA ('01)]

$$\hat{a}(t) = \frac{i}{\sqrt{\hbar}} \left[u^*(t) \hat{p} - \frac{1}{X(t)} \{ \dot{u}^*(t) - Y(t)u^*(t) \} \hat{x} \right], \quad \hat{a}^+(t) = \text{H.C.}$$

Density Operator

- The most general density operator [SPK, Page, PLB ('13)]

$$\hat{\rho}(t) = \frac{1}{Z} \exp\left[-\beta\left(\hbar\omega_0\hat{a}^+(t)\hat{a}(t) + \delta\hat{a}^+(t) + \delta^*\hat{a}(t)\right)\right]$$

- The density operator has five parameters:
 - a real constant $\beta\omega_0$ for an initial thermal distribution
 - a complex constant δ for the classical position and momentum
 - one squeeze parameter and one squeeze angle for u

Dispersion Relations

- The dispersion relation w.r.t. the density operator

$$\langle \hat{O} \rangle = \text{Tr}[\hat{O}\hat{\rho}(t)]$$

- The position and momentum expectation values

$$\langle \hat{x} \rangle = x_c(t), \quad \langle \hat{p} \rangle = p_c(t)$$

- The dispersion relations

$$\langle \hat{x}^2 \rangle - x_c^2(t) = \hbar u^* u (2\bar{n} + 1), \quad \bar{n} = \frac{1}{e^{\beta \hbar \omega_0} - 1}$$

$$\langle \hat{p}^2 \rangle - p_c^2(t) = \frac{\hbar}{X} (\dot{u}^* - Yu^*) (\dot{u} - Yu) (2\bar{n} + 1)$$

Density Matrix and Wigner Function

- The density matrix $\rho(x, x') = \langle x | \hat{\rho}(t) | x' \rangle$
- The Wigner function

$$P(x, p) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} dy \rho(x - y, x + y) e^{2ipy/\hbar} = P_T(x - x_c, p - p_c)$$

- The thermal Wigner function

$$P_T(x, p) = \frac{\tanh(\beta\hbar\omega_0/2)}{\pi\hbar} \exp\left[-\frac{\tanh(\beta\hbar\omega_0/2)}{\hbar\omega_0} H_E(x, p)\right]$$

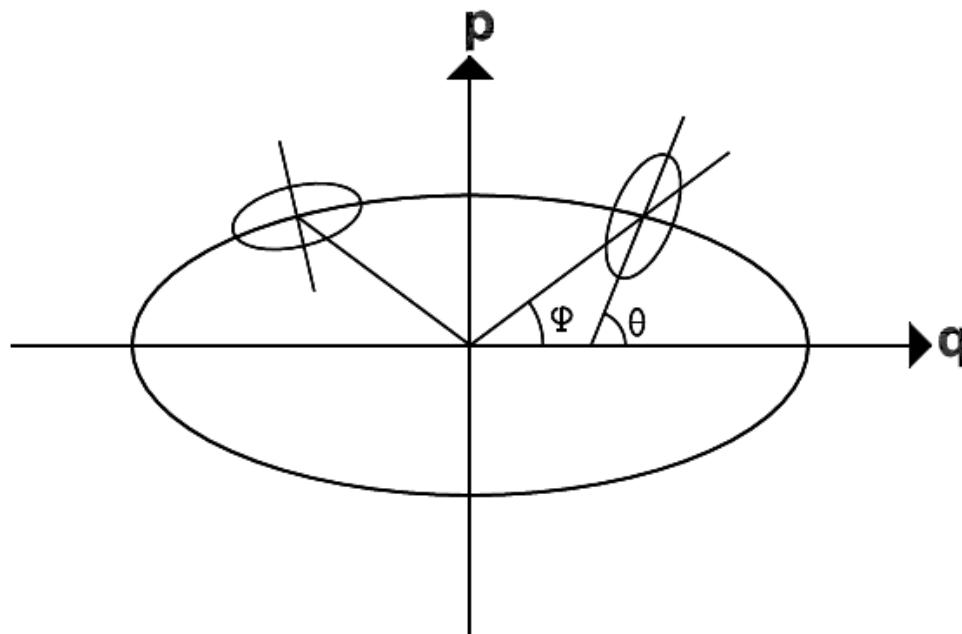
$$H_E(x, p) = \omega_0 |u|^2 \left(p - \frac{d \ln |u|}{dt} \frac{x}{X} \right)^2 + \frac{\omega_0}{4|u|^2} x^2$$

Wigner Function

- The general solution to a harmonic oscillator

$$u_{r\phi}(t) = \frac{1}{\sqrt{2m\omega_0}} [(\cosh r)e^{-i\omega_0 t} + (\sinh r e^{i\phi})e^{i\omega_0 t}]$$

- The contour of the Wigner function



Damped Oscillator Driven by External Forces

Driven Damped Oscillator

- CK damped oscillator driven by external forces

$$H(t) = H_{CK}(t) - F(t)x$$

- The classical motion is a damped oscillator

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = F(t)$$

- The external forces include gravitational waves and (thermal) noises.
- The quantum theory is still the time-dependent Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(t) = \hat{H}(t) \Psi(t)$$

Construction of Quantum States

- Two linear invariant operators [SPK, CMP ('03)]

$$\hat{b}(t) = \hat{a}_u(t) - \frac{i}{\sqrt{\hbar}} f(t) \hat{e}, \quad \hat{b}^+(t) = \text{H.C.}$$

$$\ddot{u} + \gamma \dot{u} + \omega_0^2 u = 0, \quad m e^{\gamma t} \left(u \dot{u}^* - \dot{u} u^* \right) = i$$

$$f(t) = \int_{-\infty}^t F(t') u^*(t') dt'$$

- The ground state is a coherent state of CK oscillator and has the same dispersions (minimal uncertainty)

$$\hat{b}(t) \Psi_0(x, t) = 0 \iff \hat{a}_u(t) \Psi_0(x, t) = \frac{i}{\sqrt{\hbar}} f(t) \Psi_0(x, t)$$

Displaced Number States

- The displacement operator [SPK, JKPS ('04)]

$$\hat{b}_u(t)|\alpha, 0, t\rangle = 0 \Leftrightarrow \hat{a}_u(t)|\alpha, 0, t\rangle = \alpha|\alpha, 0, t\rangle \quad \left(\alpha = \frac{i}{\sqrt{\hbar}} f(t) \right)$$

$$|\alpha, 0, t\rangle = \hat{D}_u(\alpha)|0, t\rangle, \quad \hat{D}_u(\alpha) = \exp[\alpha \hat{a}_u(t) - \alpha^* \hat{a}_u^+(t)]$$

- The displaced number states (MUS)

$$|n, t\rangle = \frac{(\hat{a}_u^+ \hat{a}_u)^n}{\sqrt{n!}} |0, t\rangle \Rightarrow |\alpha, n, t\rangle = \hat{D}_u(\alpha)|n, t\rangle$$

Centroid and Dispersions

- The centroid

$$x_c(t) = \langle \alpha, n, t | \hat{x} | \alpha, n, t \rangle = \sqrt{\hbar} [\alpha u + \alpha^* \dot{u}^*]$$

$$p_c(t) = \langle \alpha, n, t | \hat{p} | \alpha, n, t \rangle = \sqrt{\hbar m e^\gamma} [\alpha \dot{u} + \alpha^* \ddot{u}^*]$$

- The dispersion relations are the same as detector

$$\langle \alpha, n, t | \hat{x}^2 | \alpha, n, t \rangle - x_c^2(t) = \hbar u^* u (2n+1)$$

$$\langle \alpha, n, t | \hat{p}^2 | \alpha, n, t \rangle - p_c^2(t) = \hbar m^2 e^{2\gamma} \dot{u}^* \dot{u} (2n+1)$$

Conclusion

- Study the quantum states of a damped oscillator as an ideal detector for gravitational waves
 - Minimal uncertainty states (MUS)
 - Squeezed states do not beat SQL
 - Quantum invariant operators for QND measurements
- Discuss two quantum dissipation models for a damped oscillator: CK oscillator vs BFT oscillator
 - Quantum decoherence
 - Classical correlation
- Provide the Wigner function for the phase-space formulation of an ideal detector.