Cavity response in transmission

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1 Scope of this paper

This report summarizes the derivation of the transfer function from cavity displacement to the frequency fluctuation in transmission. In Advanced LIGO's Arm Length Stabilization (ALS) technique [1, 2], the transmitted light from a resonating Fabry-Pérot (FP) cavity is utilized as part of the cavity length sensor. In this scheme, one often needs to evaluate frequency fluctuations, excited by length fluctuations, in transmission of the cavity. Most of the derivation presented in this report follow the formalization done by M. Rakhmanov in his Ph.D. thesis [3].

2 Gedanken Experiment

Let us begin with a Gedanken experiment in which an FP cavity is illuminated by a laser field, the wavelength of which is λ as shown in figure 1. The optics are assumed to be well aligned and mode-matched to the laser field so that we do not consider misalignment or mode mismatch. Here, we focus on evaluating frequency fluctuations of the intracavity field E rather than that of the transmitted field E_t because they are equivalent.

The intra-cavity field propagating toward the end mirror $E(t)$ satisfies,

$$E(t) = t_1 E_{\rm in}(t) + r_1 r_2 e^{-2ik(L+\xi(t))} E\left(t - 2T\right), \tag{1}$$

where T is the one-way-trip time defined by $T = L/c$, ξ is small displacement from the resonance point[1] at a given time t and k is the wave number. Since we are interested in a resonating FP cavity, we set $2kL = 2\pi n$ with n being an integer. Thus the equation reduces to

$$E(t) = t_1 E_{\rm in}(t) + r_1 r_2 e^{-2ik\xi(t)} E\left(t - 2T\right). \tag{2}$$

[1] In general, ξ should be expressed by $\xi(t) = x_2(t-T) - x_1(t)$ for completeness. Our simplification here leads to an inaccurate time delay. See section 4 for more details

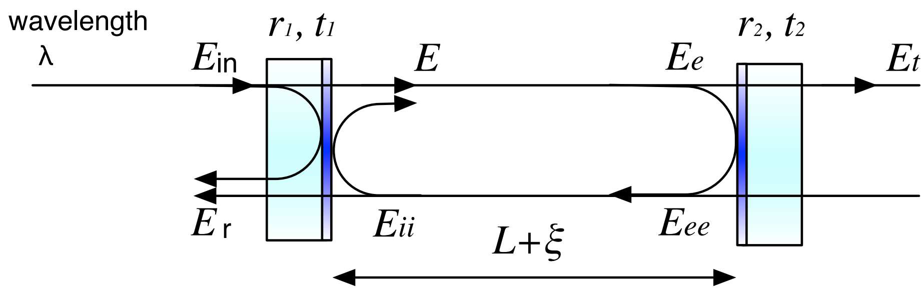

Figure 1: A Fabry-Pérot cavity and its relevant fields.

It is clear that the current intracavity field $E(t)$ is a superposition of the input pump field $E_{\rm in}(t)$ attenuated by t_a and the intracavity field retarded by the round trip time $E(t-2T)$.

3 The transfer function

Consider a small displacement so that

$$e^{-2ik\xi(t)} \approx 1 - 2ik\xi(t). \tag{3}$$

Consequently, the intra-cavity field should be also perturbed as

$$E(t) = \bar{E} + \delta E(t). \tag{4}$$

Here $\bar{E}$ is a constant value, representing the static field. On the other hand, the second term $\delta E(t)$ is a small complex field induced by cavity displacement and therefore a function of time.

Because we are not interested in any variations in the incident field, let us assume that the incident field is also a constant scalar value

$$E_{\rm in} = A. \tag{5}$$

Plunging equations (3),(4) and (5) into equation (2), one can obtain

$$\bar{E} + \delta E(t) = t_1 A + r_1 r_2 \bar{E} - 2ik\xi(t) r_1 r_2 \bar{E} + r_1 r_2 \delta E(t-2T), \tag{6}$$

where we have removed the second order perturbation term $2ikr_1r_2\xi\delta E(t-2T)$. By the way, the static solution (when no perturbation is applied) satisfies

$$\bar{E} = t_1 A + r_1 r_2 \bar{E}. \tag{7}$$

Using this relation one can remove the static terms from equation (6),

$$\delta E(t) - r_1 r_2 \delta E(t-2T) = -2ikr_1 r_2 \xi \bar{E}. \tag{8}$$

Applying the Laplace transform, we obtain

$$\delta\tilde{E} - r_1 r_2 \delta\tilde{E} e^{-2ST} = -2ikr_1 r_2 \bar{E}\tilde{\xi}. \tag{9}$$

The quantities with a tilde on top are meant to be the ones in the Laplace domain. Rearranging the equation, one can obtain,

$$\frac{\delta\tilde{E}}{\bar{E}} = -2ik\frac{r_1 r_2}{1 - r_1 r_2 e^{-2ST}}\tilde{\xi}. \tag{10}$$

This form is convenient for converting it to a phase variation of the optical field. A generic field E with a small phase deviation $\Delta\phi$ can be expressed as

$$E(\phi + \Delta\phi) \approx E(\phi)\left(1 + i\Delta\phi\right), \tag{11}$$

$$= E\left(1 + \frac{\delta E}{E}\right), \tag{12}$$

so that the small deviation in the field and phase can be related by

$$\Delta\phi = -i\frac{\delta E}{E}. \tag{13}$$

This means that equation (10) represents the phase rotation introduced by the cavity displacement ξ.

Now let us convert equation (10) into the frequency deviation of the field by applying the $\phi \to f$ formula : $\Delta\nu = \Delta\phi S/2\pi$,

$$H_x(s) \equiv \frac{\Delta\nu}{\xi} = -\frac{k}{\pi}\frac{r_1 r_2 S}{1 - r_1 r_2 e^{-2ST}}. \tag{14}$$

This is the transfer function for signals from the cavity displacement to the frequency of the transmitted light. Because this is a *Doppler effect*, the transfer function is in proportion to the velocity of cavity length motions for frequencies below the cavity pole. This consequently makes the transfer

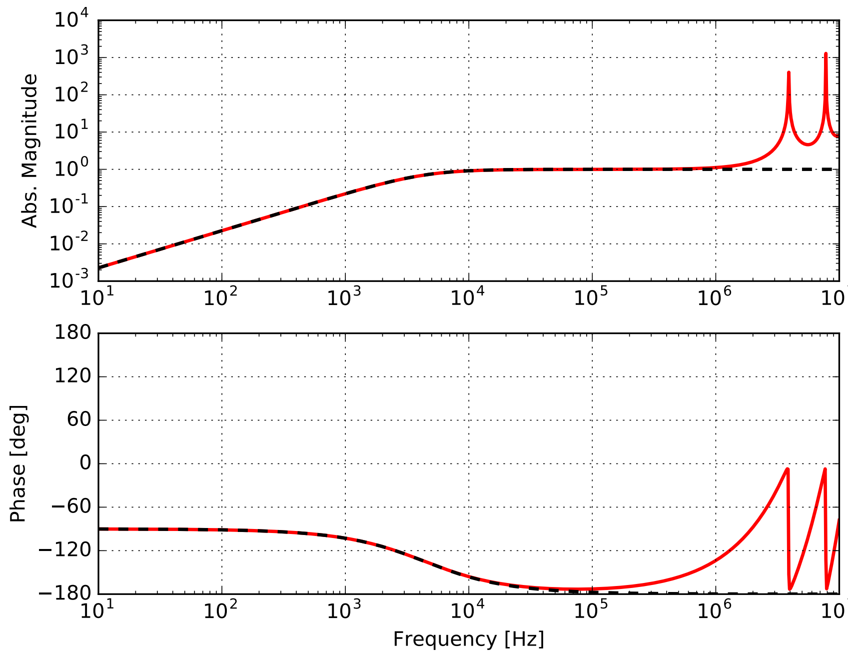

Figure 2: Bode plot of the transfer functions. The cavity pole is set to 4.4 kHz. The red curve is the one using the full expression (16) and the black dashed curve is from the approximated form (17).

function a high pass filter to cavity length. The minus sign upfront is due to the fact that the Doppler shift lowers the frequency of the field as the cavity length expands.

For practical use, it is useful to consider cavity displacement as a frequency variation in the eigen frequency of the cavity $\Delta\nu_{\rm cav}$. The eigen frequency of the cavity and displacement is connected via

$$\frac{\Delta\nu_{\rm cav}}{\nu_{\rm fsr}} = \frac{\xi}{\lambda/2}, \tag{15}$$

where $\nu_{\rm fsr}$ is the free spectral range of the cavity defined by $\nu_{\rm fsr} = c/(2L)$. Using this, one can obtain another form of the transfer function,

$$H_\nu(s) = -2ST \frac{r_1 r_2}{1 - r_1 r_2 e^{-2ST}}. \tag{16}$$

Figure 2 shows an example of this transfer function. As shown in the figure, the transfer function is a high pass filter, and becomes flat above the cavity pole, followed by periodic resonant features at high frequencies.

In most of the applications, it is handy to approximate the above equation into the linear zero-pole format by expanding the exponent term in the denominator as

$$H_\nu(s) = -\frac{S}{\omega_c + S} \quad \text{or} \quad H_\nu(\omega) = -\frac{\mathrm{i}\omega}{\omega_c + \mathrm{i}\omega}, \tag{17}$$

where ω_c is the cavity pole in units of [rad/sec] defined by

$$\omega_c = \frac{1 - r_1 r_2}{2 r_1 r_2 T}. \tag{18}$$

It is evident that as the frequency takes a larger value the magnitude of the transfer function asymptotically reaches the unity so that $|H_\nu| \to 1$.

4 Derivation in Frequency Domain

One can also derive the transfer function from a consideration purely in the frequency domain as opposed to starting from the time domain.

Suppose that the end mirror is free to move longitudinally while the position of the input mirror is fixed for simplicity. As described by Regehr in his thesis [4], the reflected light off of the oscillating end mirror with an

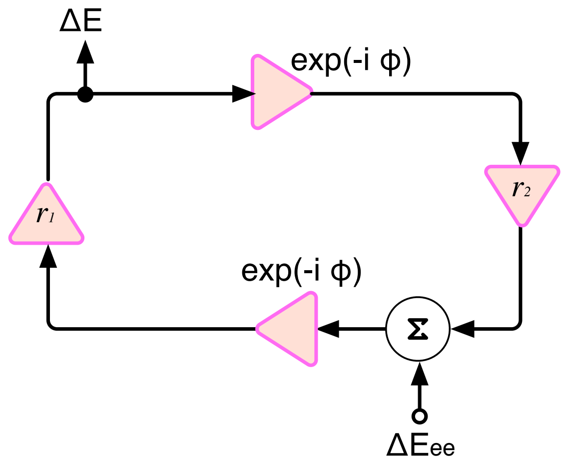

Figure 3: A block diagram for propagating perturbations.

amplitude of ξ at a frequency of ω_m produces a pair of audio sidebands around the carrier field,

$$\begin{aligned} \Delta E_\mathrm{ee} &= -2ikr_\mathrm{e}\xi\bar{E}_\mathrm{e}\cos\omega_\mathrm{m}t \\ &= -ikr_\mathrm{e}\xi\bar{E}_\mathrm{e}\left[e^{i\omega_\mathrm{m}t}+e^{-i\omega_\mathrm{m}t}\right] \end{aligned} \tag{19}$$

As usual, the first term in the square brackets represents the upper audio sideband generated by the moving end mirror whereas the other represents the lower audio sideband. In order to properly propagate this perturbed field through the system, one can fully solve the interferometer matrix as described in [4]. However, we instead perform a loop algebra which is commonly employed in classical control theories and provides more intuitive understanding for simple systems.

Think about an equivalent network depicted in figure 3. According to the loop digram, the perturbed field of the intracavity field ΔE should satisfy

$$r_1 r_2 \Delta E e^{-2i\phi} + r_1 \Delta E_\mathrm{ee} e^{-i\phi} = \Delta E, \tag{20}$$

where ϕ is the single-trip phase. Therefore, the perturbed cavity field is

$$\Delta E = \frac{r_1 e^{-i\phi}}{1 - r_1 r_2 e^{-2i\phi}}\Delta E_\mathrm{ee}. \tag{21}$$

Remember that because we are in the frequency domain, we have to distinguish between the upper sideband at $+\omega_\mathrm{m}$ and the lower sideband at $-\omega_\mathrm{m}$

when evaluating the above two expressions. In doing so, one must treat the single-trip phase as a function of the perturbation frequency

$$\phi = \frac{(\omega_0 \pm \omega_{\rm m})\, L}{c}, \tag{22}$$

where ω_0 is the main laser frequency and c is the speed of light. We consider both upper and lower sidebands at once and hence the $\pm$ sign. Letting the cavity be on resonance, one can simplify the above as

$$\phi = \pm \frac{\omega_{\rm m} L}{c}. \tag{23}$$

Plugging this into equation (21) individually for the upper and lower sidebands and then summing them, one can find

$$\begin{aligned} \frac{\Delta E}{\bar{E}} &= -ik\xi \left[\frac{r_1 r_2 e^{-i\omega_{\rm m} L/c}}{1 - r_1 r_2 e^{-2i\omega_{\rm m} L/c}} e^{i\omega_{\rm m} t} + ({\rm c.c.}) \right]. \\ &= -2ik\xi {\rm Re} \left[\frac{r_1 r_2 e^{-i\omega_{\rm m} L/c}}{1 - r_1 r_2 e^{-2i\omega_{\rm m} L/c}} e^{i\omega_{\rm m} t} \right]. \end{aligned} \tag{24}$$

where (c.c.) stands for the complex conjugate of its previous term and we have used the relation $\bar{E} = \bar{E_{\rm cc}}$.

As described in [4], the conversion of the above to a generic transfer function can be done by removing the Re operation and the $\exp(i\omega_{\rm m} t)$ term. Therefore,

$$\frac{\Delta E / \bar{E}}{\xi} = -2ik \frac{r_1 r_2 e^{-i\omega_{\rm m} L/c}}{1 - r_1 r_2 e^{-2i\omega_{\rm m} L/c}} \tag{25}$$

which is identical to equation (10) except that now it includes a delay term $\exp(-i\omega L/c)$ precisely due to the fact that we have assumed a certain perturbation i.e. displacement on the end test mass.

References

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