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Uncertainty Relation and GW Interferometer

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Introduction

In 1980's there were hot debates on the possibility of beating the standard quantum limit among Caves, Yuen and Ozawa.

I would like to briefly look at the history and convince you that the uncertainty principle for the point mass is actually irrelevant for the present day LIGO/KAGRA type interferometer.

Very naively the tiny displacement of mirror ≈ nuclear size by an incident GW in the Michelson-type interferometer suggests possible relevance of uncertainty principle.

Caves et al. once claimed that the power spectrum of the GW $h(t)$

$$S(\omega) := \int dt \, \langle h(t) h(0) \rangle e^{i\omega t}$$

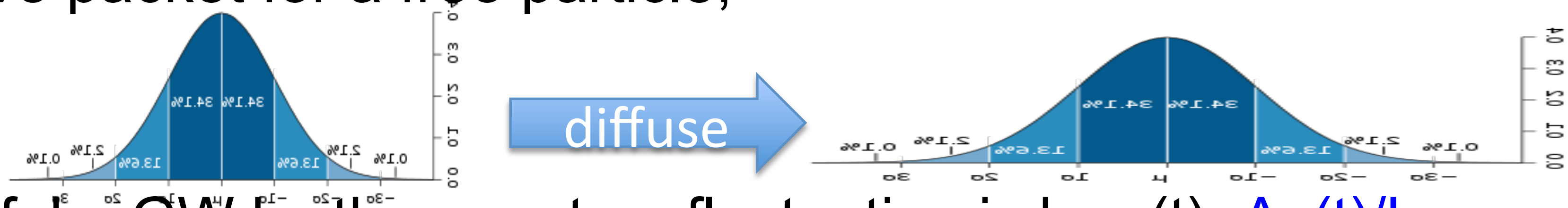
is bounded by the standard quantum limit

$$S^{SQL}(\omega) \approx \hbar / m\omega^2 L^2$$ (m:mirror mass, L:arm length)

The reasoning might be the following.
Let the quantum fluctuation of a point mass(=mirror) be $\Delta x(t)$. Then the uncertainty principle would imply

$$<\Delta x(t)\, \Delta x(0)> \;\; \geq \hbar |t|/m \qquad (\#)$$

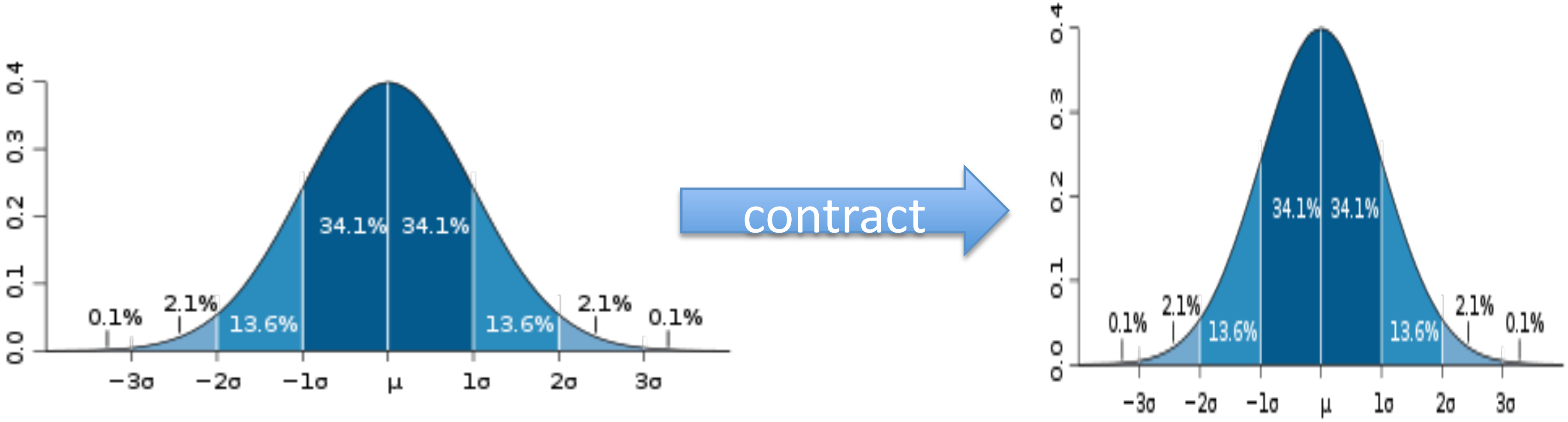
The RHS essentially comes from the broadening of wave packet for a free particle,



The fake GW by the quantum fluctuation is $h_{fake}(t)=\Delta x(t)/L$ with L being the arm length. We would have

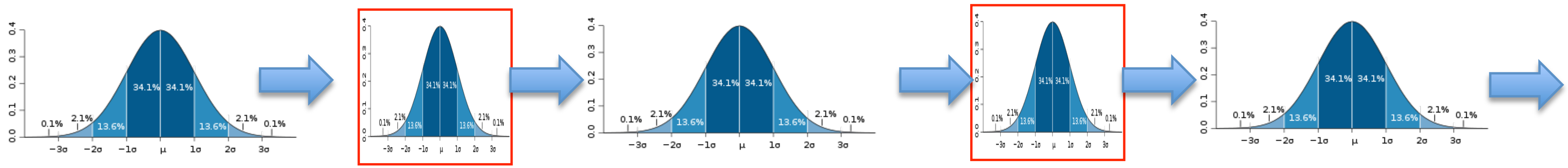
$$S(\omega):= \int dt\, e^{i\omega t} < h_{fake}(t) h_{fake}(0) > \geq \int dt\, e^{i\omega t}\, \hbar |t|/mL^2$$
$$\approx \hbar/m\omega^2 L^2$$

However, as Yuen observed ,(#) does not hold for a “contracting wave packet”.



The contracting wave packet can be realized e.g., a non-miniimal gaussian wave packet in the harmonic potential. Actually it repeats the diffusion and contraction periadically.

If we observe the point mass at the timings of contraction, the breadth $\Delta x(t)$ will become smaller than the initial one.



observe x　　　　observe x

Later Ozawa explicitly showed an example of interaction Hamiltonian for the measurement which gives zero error if the timing is fine tuned.

Caves et al. Rev.Mod.Phys.52 341(1980)
Yuen, Phys.Rev.Lett 51 719 (1983)
Caves, Phys.Rev.Lett 54 2465(1985)
Ozawa, Phys.Rev.Lett 60 385 (1988)

The debates in 1980's concluded that SQL can be overcome.

Maddox, Nature 331 559 (1988)

However, after all the GW is a classical signal. How the uncertainty principle is relevant at all?

Actually this intuition turned out to be correct.

Braginsky et al. showed that the initial position and momentum of a point mass can be eliminated from the data sequence either by filtering or signal recycling.

Braginsky et al. Phys.Rev.D67 082001(2003)

2. Filtering

Consider a measurement Hamiltonian,

$$H= p^2/2m-F(t)x -\Sigma_0^{N-1} xP_r\delta(t-\tau r)$$

Here (x,p) is the position and momentum of a point mass. P_r is the momentum conjugate to the position Q_r of the r-th detector impulsively coupled to the Position of the point mass at time $t=\tau r$, $r=0,1,2,\ldots.N-1$.
Later we identify (Q_r,P_r) with the quadratures of the electromagnetic fields in the laser interferometer.
$F(t)$ is the external force corresponding to GW.

The solution for $x_r(t)$ just after τr is given by

$$x_r = x_0 + r\tau p_0/m + \Sigma_0^r P_s(r-s)\,\tau/m + \xi_r$$

where ξ_r is a function of the external force $F(t)$.
At that time $t=\tau r$ the detector position Q_r is

$$Q_r = Q_r^{before} - x_r$$
$$= Q_r^{before} - x_0 - r\tau p_0/m - P_r^{before} r\tau/m - \Sigma_1^r P_s(r-s)\,\tau/m - \xi_r$$

Since the terms which contain (x_0, p_0) are linear in r, we can eliminate them by composing a filter corresponding to a discrete version of the second derivative: $Q_{r+1} - 2Q_r + Q_{r-1}$

$$Q_2 = Q_2^{before} - x_0 - 2\tau p_0/m - 2P^{before}\tau/m - P_1\tau/m - \xi_2$$

$$Q_1 = Q_1^{before} - x_0 - \tau p_0/m - P^{before}\tau/m - \xi_1$$

$$Q_0 = Q_0^{before} - x_0 - \xi_0$$

The combination

$$Q_2 - 2Q_1 + Q_0 = Q_2^{before} - [2Q_1^{before} + P_1^{before}\tau/m] + Q_0^{before} + \xi_2 - 2\xi_1 + \xi_0$$

eliminates the initial position x_0 and momentum p_0.

For example,

$$Q_2 - 2Q_1 + Q_0 = Q_2^{before} - [2Q_1^{before} + P_1^{before}\tau/m] + Q_0^{before} + \xi_2 - 2\xi_1 + \xi_0$$

If the initial state is chosen as an eigenstate of Q_2^{before} and Q_0^{before} and further of

$$2Q_1^{before} + P_1^{before}\tau/m$$

that is, a squeezed state, the data sequence $Q_2 - 2Q_1 + Q_0$ contains only the classical information of GW.

The important point is

$$Q_r = Q^{before} - x_0 - r\tau p_0/m - P^{before} r\tau/m - \Sigma_1^{\,r} P_s (r-s)\,\tau/m - \xi_r$$

are commutable,

$$[Q_r, Q_s] = 0$$

The contribution from (x_0,p_0) of the mass point and from (Q^{before},P^{before}) of the detector exactly cancel out.

so that measurement of Q_r's are compatible.

Note that

$$[P_r, P_s] = 0,$$

since $P_r = P^{before}$.

In a laser interferometer, the electromagnetic field is the detector variable

$$E = E_{in} + iC(t)x + \Sigma_r \ldots .[a_r e^{-i\omega t} - a^*_r e^{i\omega}]$$

The second term comes from the phase shift by the position of the mirror. a_r and a^*_r are annihilation and creation operators.

The correspondence to the previous model is

$$P_r \Leftrightarrow a_r + a^*_r$$

$$Q_r \Leftrightarrow a_r - a^*_r$$

The actual measurement in the GW interferometer is the photon number counting $N(t)$ or N_r

They are also commutable, since $N_r \propto (P_r)^2 + (Q_r)^2$

so that the measurement of N_r does not disturb the subsequent measurements of N_s's.

It seems that we can always construct a “filtering” to eliminate the initial position and momentum of a point mass if the measuring interaction is bilinear as far as the detector degrees of freedom (electromagnetic field) is much larger than that of the mass point (mirrors).

As the other method to eliminate the the initial position and momentum of a point mass, one may use the damping of the mode e.g., by signal recycling.

This kind of device can evade the uncertainty principle, because it only measures the external classical force.

However, is this the only way?

We will show a way to beat the SQL by directly measuring Q_r in what follows.

3. Ozawa's Uncertainty Relation

Suppose we do not use the filtering discussed in the previous slide but directly measure Q_r

Is SQL unavoidable?

I claim that the answer is NO.
There remains a way to overcome the uncertainty relation.

Ozawa reformulated Heisenberg's uncertainty principle, on the basis of rigorous measurement theory of the Completely Positive (CP) map.

$$\varepsilon(A)\,\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\,\eta(B) \geq |\langle\psi|[A,B]|\psi\rangle|/2$$

where ε(A) is the error in the measurement of A, η(B) is the disturbance of B by the measurement of A.
σ (A) and σ(B) are the quantum fluctuations of A and B in the state |ψ> i.e., the standard deviations.

M.Ozawa: Phys.Rev. A67,042105 (2003)

Here $A=Q_r$ and $B=P_r$

Is it possible for ε(Qr)=0?
The original Heisenberg's uncertainty relation

$$\varepsilon(Q_r)\,\eta(P_r) \geq \hbar/2 \qquad (\#\#)$$

would imply $\eta(P_r) = \infty$ and so is $\eta(N_r)$.
However, actually (##) is NOT correct.
Instead we have

$$\sigma(Q_r)\,\eta(P_r) \geq \hbar/2$$

The disturbance $\eta(P_r)$ can be finite!

4. Summary

The standard quantum limit （SQL）can be avoided either by filtering or signal recycling, which eliminate the initial position and momentum of a mass point. (Braginsky et al.)

It is also possible to directly beat SQL choosing the initial state and measurement apparatus in principle. (Yuen, Ozawa)

The guide line can be found in Ozawa’s inequality.

Thank you for your attention!