3D Models of Vibration Isolation Systems in Mathematica

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Concept

- Three-dimensional rigid-body models in Mathematica
- Inspired by Mark Barton's suspension models
  http://www.ligo.caltech.edu/~e2e/SUSmodels/
- The model can provide eigenfrequencies and eigenmodes of the system (with 3D Graphics), transfer functions from disp to disp, and from force/torque to disp of every 6 DOFs of the given objects.
- Asymmetry in geometry, wire length, stiffness, and etc. may be taken into account.
- The simulation provides state-space matrices of the system and exports them to a MATLAB code, for studying active controls and etc.

Method of Calculation

(1) Express the potential energy E_P, kinetic energy E_K and damping energy E_D of the system in terms of the coordinates and coordinate velocities.

$$E_P = E_P(x_1,\cdots,x_n),\quad E_K = E_K(x_1,\cdots,x_n,\dot{x}_1,\cdots,\dot{x}_n),\quad E_D = E_D(\dot{x}_1,\cdots,\dot{x}_n)$$

(2) Minimize the potential energy E_P to find the equilibrium points of the system.

$$\vec{x}_{(eq)} = (x_{1(eq)},\cdots,x_{n(eq)})$$

(3) Differentiate the potential energy of the system with respect to pairs of coordinates at equilibrium to create the stiffness matrix K. In a similar way, differentiate the kinetic energy and damping energy with respect to coordinate velocities to create the mass matrix M and damping matrix G.

$$K_{ij} = \left.\frac{\partial E_P}{\partial x_i \partial x_j}\right|_{x=x_{(eq)}},\quad M_{ij} = \left.\frac{\partial E_K}{\partial \dot{x}_i \partial \dot{x}_j}\right|_{x=x_{(eq)}},\ G_{ij} = \left.\frac{\partial E_D}{\partial \dot{x}_i \partial \dot{x}_j}\right|_{x=x_{(eq)}}$$

(4) Do diagonalization of the stiffness and mass matrices to obtain the eigenfrequencies and eigenmodes of the system.

$$(M^{-1}K)\vec{e}_i = \omega_i^{\ 2}\vec{e}_i$$

(5) Calculate transfer functions of the system from the stiffness, mass and damping matrices.

More details are in LIGO-T020205-02-D ("Models of the Advanced LIGO Suspensions in Mathematica" by Mark Barton).

Potential Terms

Wires

Assume a wire as a massless beam, with a spring constant defined by its young's modulus, length and diameter. The potential energy of each wire is broken up in two terms, longitudinal stretching energy and torsional energy.

$$E_{stretch} = \frac{1}{2}(l(x) - l_0)^2, \qquad E_{torsion} = \frac{1}{2}\frac{GJ}{l_0}\Delta\theta_{torsion}$$

Springs (GAS Filters)

A spring is taken to join two points in different objects and apply restoring forces and pre-load forces to them. The potential energy is given by:

$$E_{spring} = \frac{1}{2}(\Delta x, \Delta y, \Delta z, \Delta Pitch, \Delta Yaw, \Delta Roll)K_{spring}\begin{pmatrix}\Delta x\\ \Delta y\\ \Delta z\\ \Delta Pitch\\ \Delta Yaw\\ \Delta Roll\end{pmatrix} + f_{preload}\begin{pmatrix}\Delta x\\ \Delta y\\ \Delta z\\ \Delta Pitch\\ \Delta Yaw\\ \Delta Roll\end{pmatrix}.$$

Coordinate System

(VIRGO's Reference)

X-axis: Transversal, Y-axis: Vertical, **Z-axis: Longitudinal** (Beam-axis)

Pitch: Rotation around x-axis, Yaw: Rotation around y-axis, Roll: Rotation around z-axis

Demonstration

1. Simple Pendulum (Suspending LIGO mirror with four 50-cm wires)

Parameters

Body (Test Mass)

mTM=10.7kg;(*Mass*)
ixTM=0.051kg meter^2;(*MOI around x-axis*)
iyTM=0.051kg meter^2;(*MOI around y-axis*)
izTM=0.078kg meter^2;(*MOI around z-axis*)
iTM={{ixTM,0,0},{0,iyTM,0},{0,0,izTM}};(*MOI Tensor*)
rTM=12.5cm;(*radius*)
lTM=10cm;(*length*)
shapeTM="Cylinder";(*Body Shape*)

Geometry

dzTM=3cm;(*z-separation of wires *)
dxTM=25cm;(*x-separation of wires *)
dyTM=1mm;(*height of wire break-off above COM at TM*)

Wires

lNwire={50,50,50,50}cm;(*Natural length*)
dwire={0.20,0.20,0.20,0.20}mm;(*Diameter*)
matwire={"W","W","W","W"};(*Material*)

The attachment points for wires (in local body coordinate)

clampu={{-dxu/2,0,+dzu/2},{+dxu/2,0,+dzu/2},{-dxu/2,0,-dzu/2},{+dxu/2,0,-dzu/2}};
clampl={{-dxl/2,dyTM,+dzl/2},{+dxl/2,dyTM,+dzl/2},{-dxl/2,dyTM,-dzl/2},{+dxl/2,dyTM,-dzl/2}};

Initial Position & Orientation

lw=50cm;(*typical wire length*)
initg={0,0,0,0,0,0};(*Ground*)
initTM={0,-lw,0,0,0,0};(*Test Mass*)

Equilibrium Point of the System

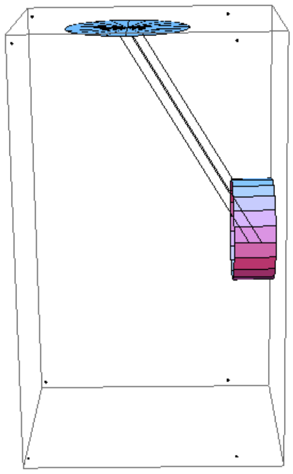
{xTM→0.,yTM→-0.502016,zTM→0.,pitchTM→0.,yawTM→0.,rollTM→0.}

Eigen Mode List

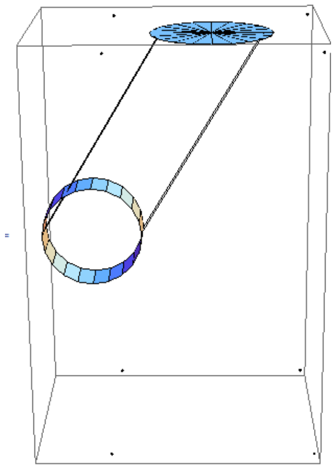
{N, Freq, Type(Amplitude)},
{1, 0.70425, zTM(-0.999956)},
{2, 0.704253, xTM(-1.)},
{3, 1.28425, yawTM(1.)},
{4, 3.40524, pitchTM(-1.)},
{5, 15.6376, yTM(1.)},
{6, 22.8949, rollTM(-1.)}

Eigen Mode 3D-Graphics

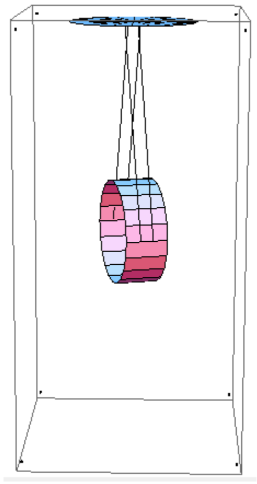
N=1(pendulum mode in Z)



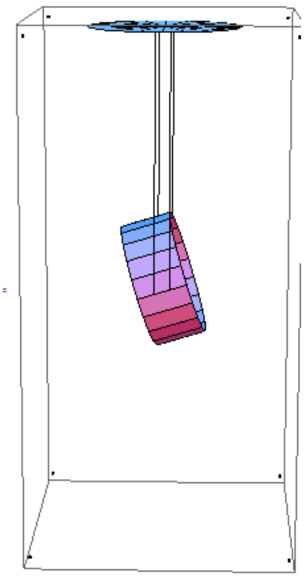
N=2 (pendulum mode in X)



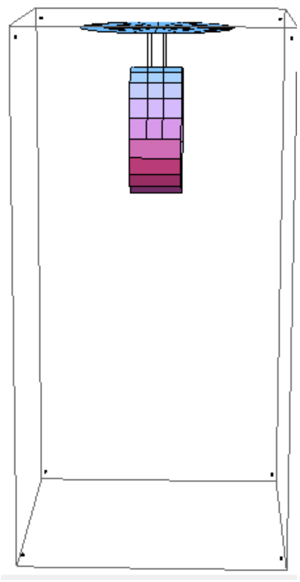
N=3 (yaw mode)



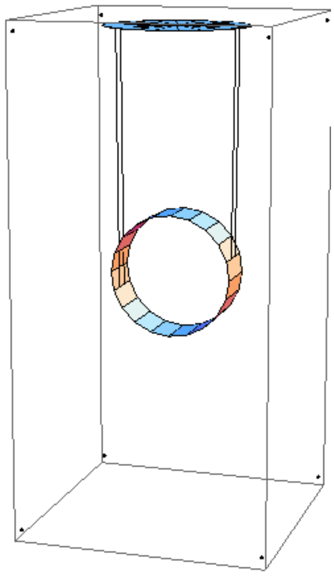
N=4 (pitch mode)



N=5 (vertical bounce)

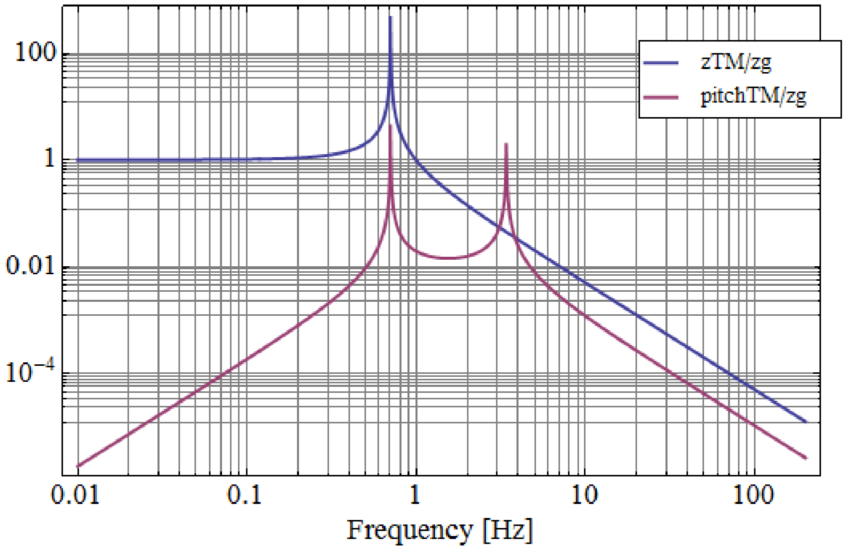


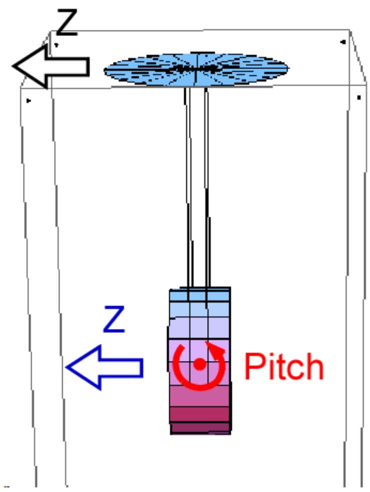
N=6 (roll mode)



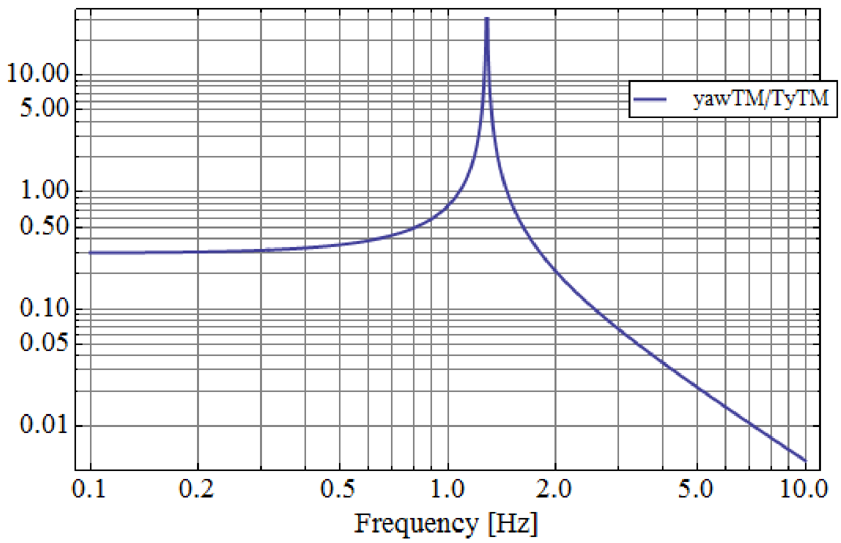
Transfer Function Plot

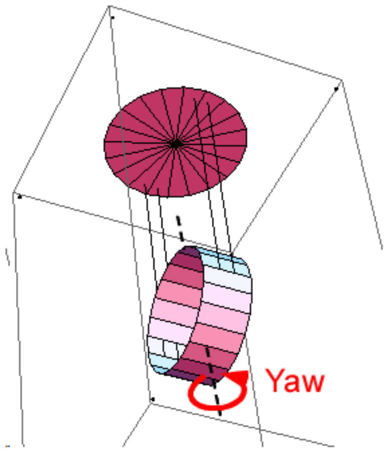
Ground displacement (Z) → TM displacement (Z and Pitch)





Torque exerted to TM (Yaw) → TM displacement (Yaw)





With asymmetry in wire length

lNwire = {50, 50, 50, 50.3} cm;(*Natural length*)

Equilibrium Point

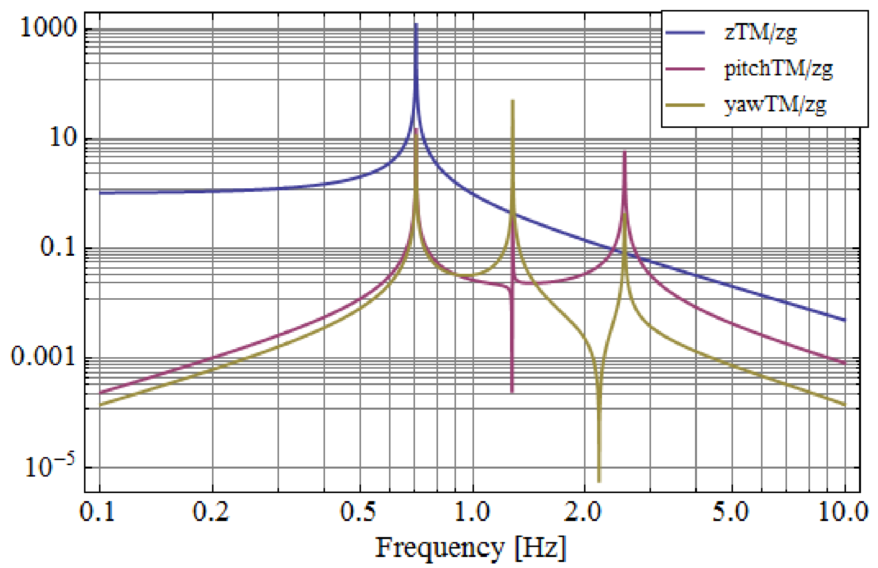
{xTM→-1.77042×10^{-6},yTM→-0.502051,zTM→0.00001465, pitchTM→-0.016602,yawTM→-2.71059×10^{-6},rollTM→-0.00200601}

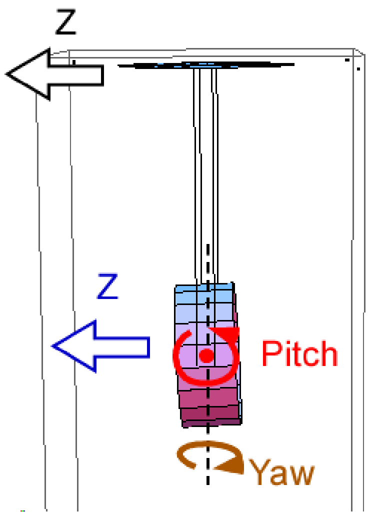
Eigen Mode List

N	Freq	Type
1	0.704413	zTM 0.997505
2	0.704417	xTM 0.99763
3	1.28242	yawTM 0.999715
4	2.56178	pitchTM 0.997323
5	11.7401	yTM 0.999695
6	16.3101	rollTM 0.996353

Transfer Function Plot

Ground Displacement (Z) → TM displacement (Z, Yaw and Pitch)





(Yaw mode is coupled with pendulum mode)

* As the asymmetry of the wire (3 mm) is larger than wire stretching length (~2 mm), some wires may be loose in the real system. However, in this simulation, wires are regarded as ideal springs, or more specifically, they work as springs even when they are shorter than natural length.

2. Double Pendulum Model (LCGT Payload, IM+TM+RM)

Parameters

Bodies

(* IM, TM, RM, *)
mass = {80.0, 10.7, 90.0} kg;(*mass*)
moix = {1.20, .051, 4.00} kg meter^2; (*moment of inertia around x-axis*)
moiy = {2.40,.051, 4.00} kg meter^2; (*moment of inertia around y-axis*)
moiz = {1.20,.084, 8.00} kg meter^2; (*moment of inertia around z-axis*)

Wire

(* IM, TM, RM, *)
matw = {"MS", "W", "C70"};(*material*)
lNw = { {2.084},{2.084, 2.084, 2.084, 2.084},{2.084, 2.084, 2.084, 2.084}} meter; (*natural length*)
dw = { {2.10},{0.15, 0.15, 0.15, 0.15},{0.70, 0.70, 0.70, 0.70}} mm; (*diameter*)
dyu = {{0.00},{0.00, 0.00, 0.00, 0.00},{0.00, 0.00, 0.00, 0.00}} mm; (*height of wire break-off below COM*)
dyl = {{-3.0},{1.00, 1.00, 1.00, 1.00},{1.00, 1.00, 1.00, 1.00}} mm; (*height of wire break-off above COM*)
dx = {0.00, 25.0, 34.0} cm;(*x-separation of wires*)
dz = {0.00, 3.00, 6.00} cm;(*z-separation of wires*)
nw = {1, 4, 4};(*number of wires*)

GAS Filter

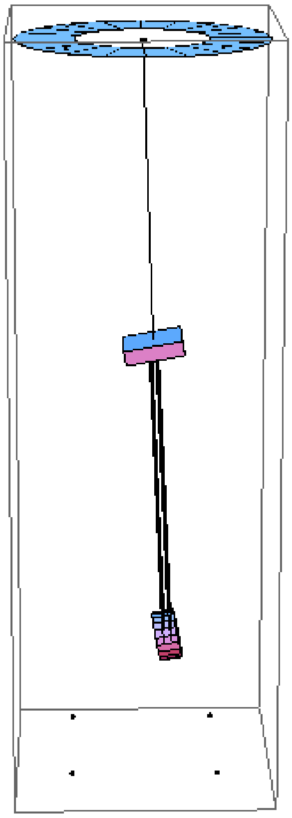
f0GAS={300}mHz;(*resonant frequency*)
phiGAS={0.01};(*loss angle*)
wdGAS={{0,1,0}};(*working direction*)

Eigen Mode List

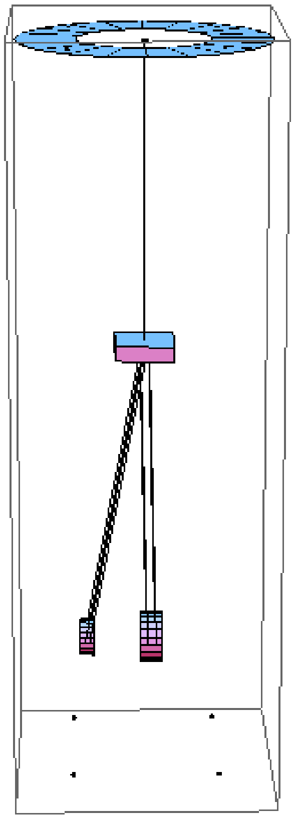
{N, Freq, Type, , , , },
{1, 0.0161036, yawIM(0.576599), yawTM(0.576977), yawRM(0.578473), , },
{2, 0.223471, rollIM(0.576258), rollTM(0.576524), rollRM(0.577891), , },
{3, 0.259871, pitchIM(0.439005), zTM(-0.418244), pitchTM(0.451518), zRM(-0.422052)},
{4, 0.262844, rollIM(-0.337478), xTM(0.552851), rollTM(-0.337686), xRM(0.545749), rollRM(-0.338786)},
{5, 0.293877, pitchIM(-0.554785), pitchTM(-0.57433), pitchRM(-0.599832), , },
{6, 0.299516, yIM(-0.576436), yTM(-0.578006), yRM(-0.577608), , },
{7, 0.344863, xTM(-0.992746), , , , },
{8, 0.344886, zTM(0.990168), , , , },
{9, 0.456134, yawIM(-0.41356), yawTM(-0.873029), , , },
{10, 0.639907, yawTM(-0.999322), , , , },
{11, 0.687048, zIM(0.902068), zTM(-0.30365), zRM(-0.305064), , },
{12, 0.687064, xIM(-0.90234), xTM(0.303722), xRM(0.305141), , },
{13, 1.25447, pitchTM(-0.999818), , , , },
{14, 2.0554, pitchIM(0.838134), pitchTM(-0.49613), , , },
{15, 5.8127, yTM(0.99493), , , , },
{16, 7.83234, rollTM(-0.997561), , , , },
{17, 9.88554, yIM(0.71734), yTM(-0.366096), yRM(-0.592787), , },
{18, 10.9316, rollIM(0.630762), rollTM(-0.771166), , , }

Eigen Mode 3D Graphics

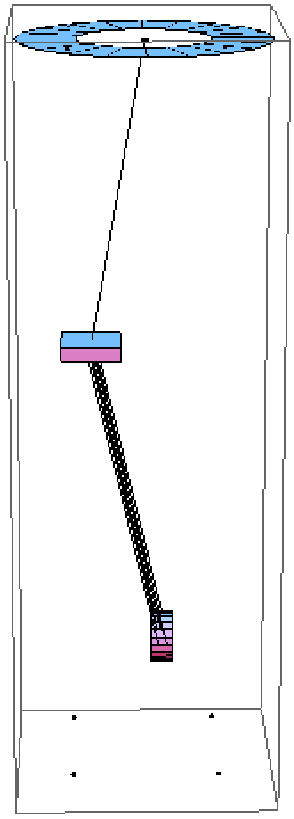
N=2 (Z and pitch)



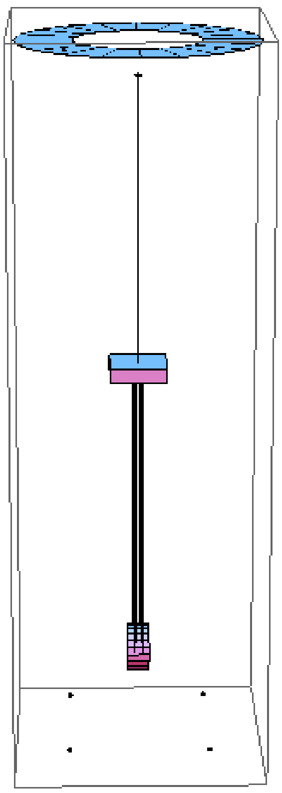
N=8 (Test Mass Z)



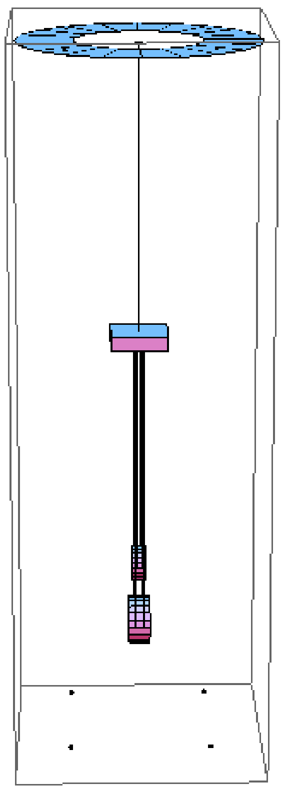
N=15 (pendulum mode X)



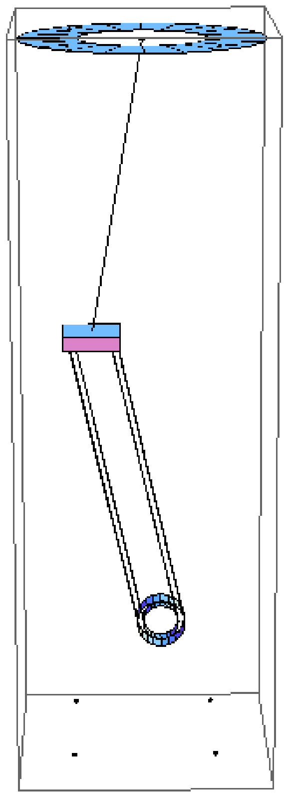
N=6 (GAS Filter Y)



N=15 (Test Mass Y)

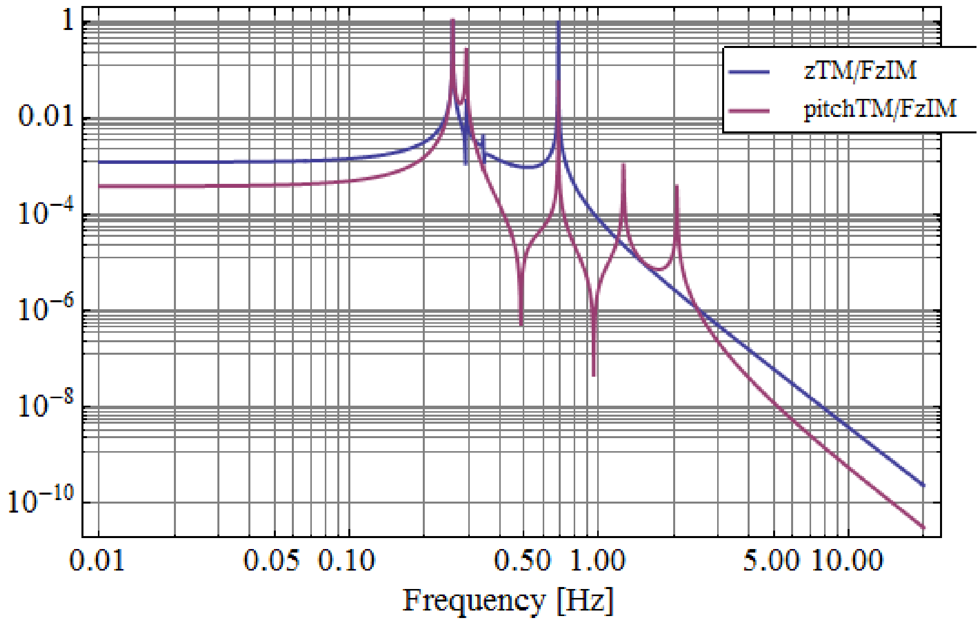


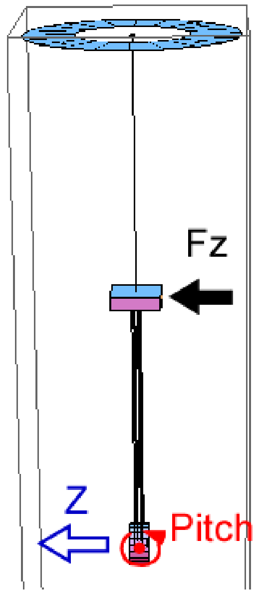
N=12 (pendulum mode X)



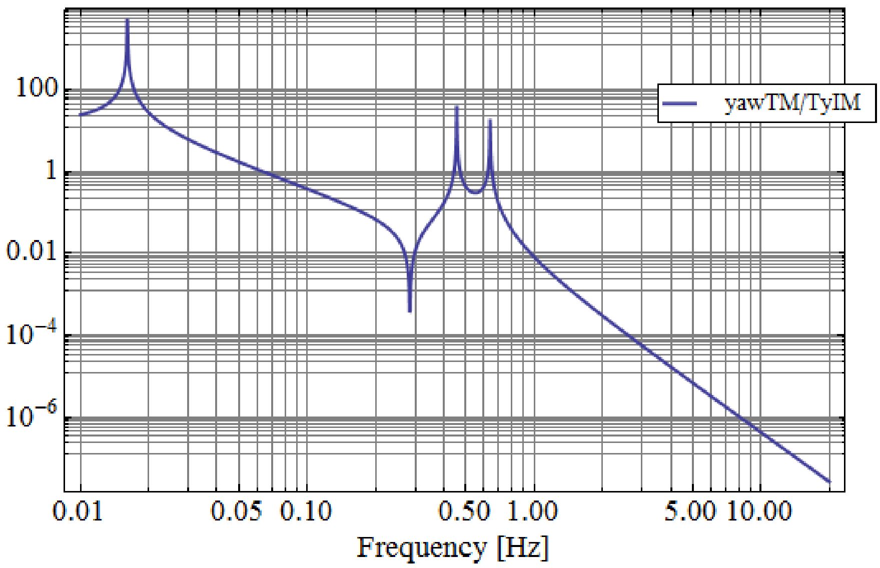
Transfer Function Plots

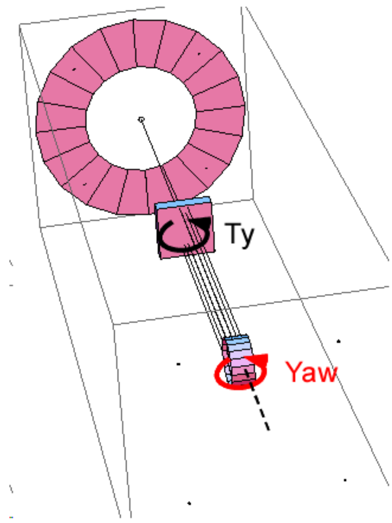
Force exerted to IM (Z) → TM displacement (Z and pitch)





Torque exerted to IM (Yaw) → TM displacement (Yaw)

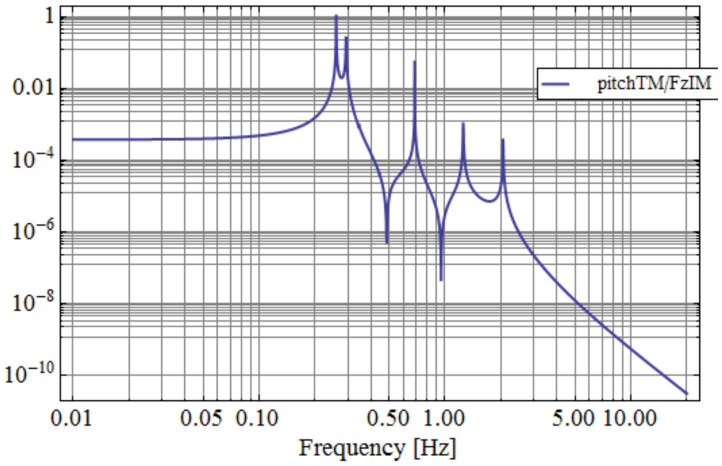


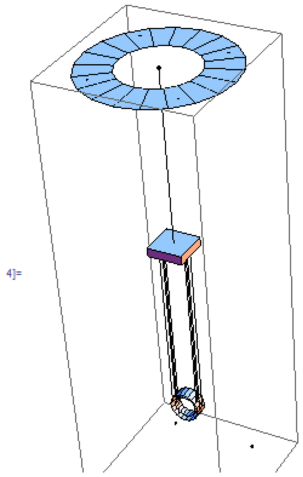


Comparison with Other Models

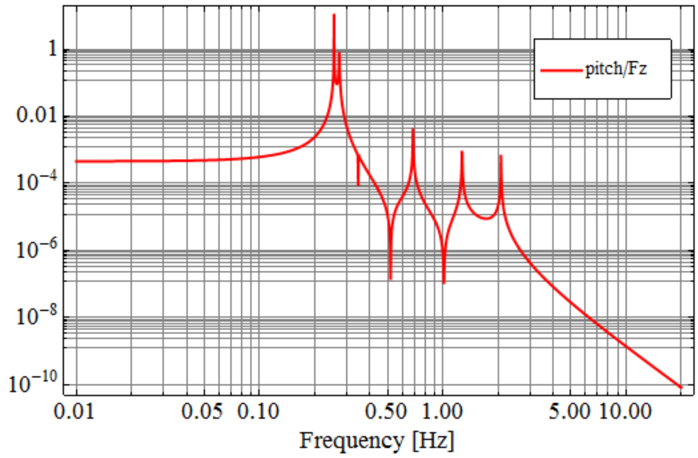
Transfer function [force exerted to IM (Z) → TM displacement (pitch)]

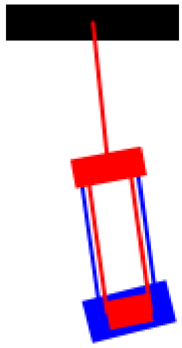
Mathematica 3D Model (this model)



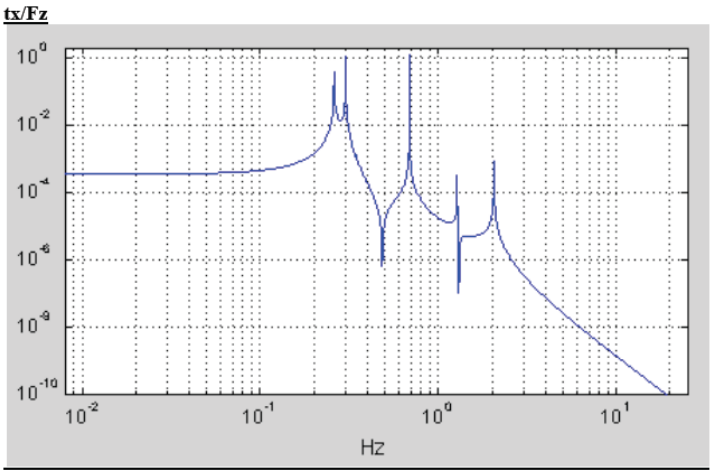


Mathematica 2D Model (conventional model)





MATLAB 3D Model (provided by E. Majorana)

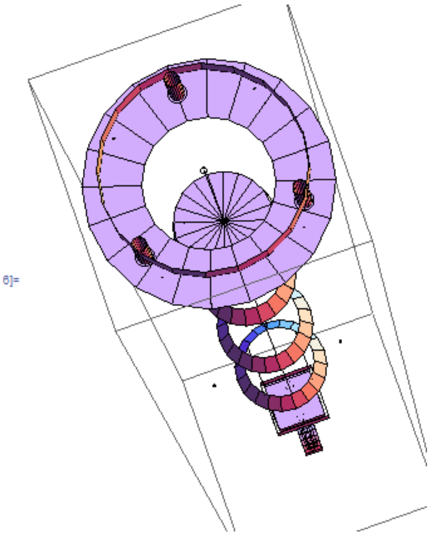


Good agreement with other models

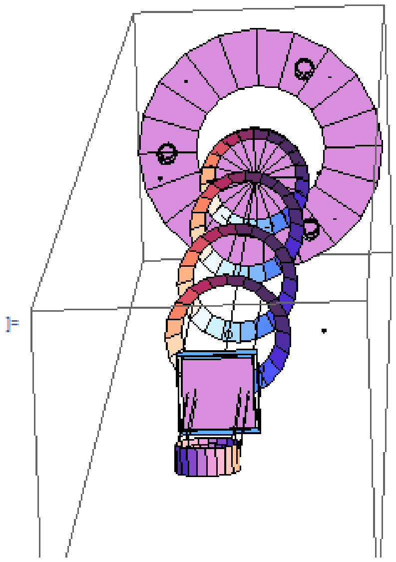
(although there are still small differences in zero-point frequencies)

3. Type-A VIS Full 3D Model

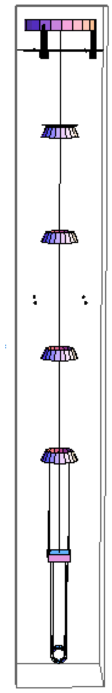
3D Graphics



Top View



Bottom View



Side View

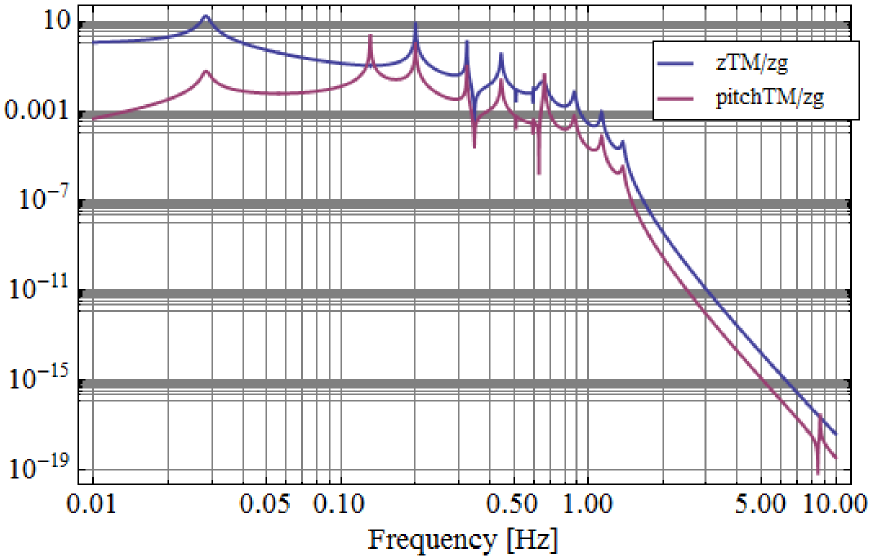
Eigen Modes

{N, Freq, Type (Amplitude), , , , , , , , },
{1, 0.00551657, yawF4(-0.388889), yawIM(-0.440602), yawIRM(-0.388999), yawTM(-0.442156)},
{2, 0.0139331, yawIM(0.51093), yawTM(0.522649), yawRM(0.510958), , , , , , },
{3, 0.0189962, yawIM(-0.526981), yawTM(-0.5499), yawRM(-0.527036), , , , , , },
{4, 0.0240614, yawF1(0.431616), yawF3(-0.635662), yawF4(0.300168), yawIRM(0.301795), , , , , },
{5, 0.0284098, xF0(-0.324966), xF1(-0.327245), xF2(-0.32953), xF3(-0.332112), xF4(-0.334411)},
{6, 0.0284098, zF0(0.324966), zF1(0.327245), zF2(0.32953), zF3(0.332112), zF4(0.334411)},
{7, 0.0329221, yawF1(-0.742689), yawF2(0.627233), , , , , , , },
{8, 0.0967754, yawF0(0.998618), , , , , , , , },
{9, 0.131343, pitchTM(0.978039), , , , , , , , },
{10, 0.132422, yF4(0.360243), yIM(0.447665), yIRM(0.360355), yTM(0.447887)},
{11, 0.152369, yawIM(-0.649954), yawTM(0.386547), yawRM(-0.654318), , , , , , },
{12, 0.19955, rollIM(0.344688), xTM(0.448604), rollTM(0.355742), xRM(0.450667)},
{13, 0.200027, zIM(0.370713), zIRM(0.318336), zTM(0.558706), zRM(0.558443), , , , , },
{14, 0.2213, rollIM(0.569554), rollTM(0.592182), rollRM(0.5689), , , , , , },
{15, 0.322592, zIRM(-0.893323), , , , , , , , },
{16, 0.322854, xIRM(-0.892765), , , , , , , , },
{17, 0.344872, zTM(-0.993559), , , , , , , , },
{18, 0.344897, xTM(-0.993695), , , , , , , , },
{19, 0.352971, yawIRM(0.987389), , , , , , , , },
{20, 0.360329, yF1(0.368838), yF2(0.452462), yF3(0.403325), yIM(-0.381287), yTM(-0.382693)},
{21, 0.407931, pitchF4(0.705728), pitchIRM(0.70784), , , , , , , },

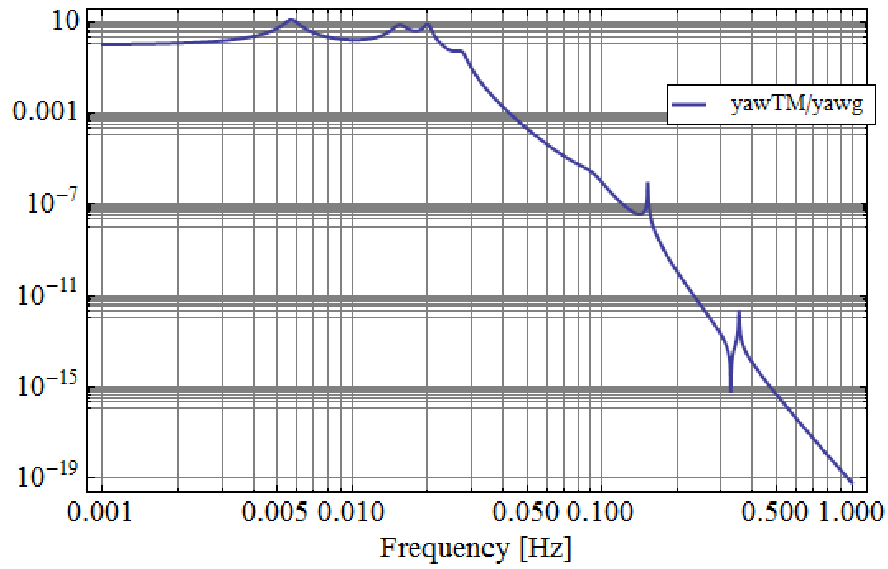
{22, 0.407931, rollF4(0.705728), rollIRM(0.70784), , , , , , , },
{23, 0.44359, zF3(0.356593), zF4(0.333768), zIRM(-0.515477), zTM(-0.306568)},
{24, 0.443857, xF3(0.369265), xF4(0.345214), xIRM(-0.53153), xTM(-0.316692)},
{25, 0.509417, pitchF3(-0.998981), , , , , , , , },
{26, 0.509417, rollF3(-0.998982), , , , , , , , },
{27, 0.590965, yF1(-0.433426), yF2(-0.30071), yF4(0.544589), yIRM(0.547984), , , , , },
{28, 0.59801, pitchF2(-0.998027), , , , , , , , },
{29, 0.598011, rollF2(-0.998147), , , , , , , , },
{30, 0.636102, pitchF1(0.933133), , , , , , , , },
{31, 0.636106, rollF1(-0.934035), , , , , , , , },
{32, 0.662865, pitchIM(0.711116), pitchRM(0.700528), , , , , , , },
{33, 0.668016, rollF1(-0.924881), , , , , , , , },
{34, 0.668393, pitchF1(0.502808), pitchIM(-0.596476), pitchRM(-0.589653), , , , , , },
{35, 0.87652, xF1(0.397203), xF2(0.415695), xF4(-0.550891), xIM(0.462687), , , , , },
{36, 0.876727, zF1(-0.369842), zF2(-0.386769), zF4(0.512557), zIM(-0.430623), , , , , },
{37, 0.881129, yF1(-0.504081), yF3(0.659678), yF4(-0.36739), yIRM(-0.372521), , , , , },
{38, 1.1322, xF1(-0.448983), xF3(0.627541), xF4(-0.588851), , , , , , },
{39, 1.13225, zF1(-0.444658), zF3(0.621458), zF4(-0.583422), , , , , , },
{40, 1.17472, yF1(0.413324), yF2(-0.748044), yF3(0.495142), , , , , , },
{41, 1.3788, xF1(-0.593646), xF2(0.686852), xF3(-0.381003), , , , , , },
{42, 1.3788, zF1(-0.593533), zF2(0.686725), zF3(-0.380943), , , , , , },
{43, 1.88551, yawRM(-0.999772), , , , , , , , },
{44, 3.08856, rollIM(0.698835), rollRM(0.707089), , , , , , , },
{45, 5.99688, yTM(-0.995326), , , , , , , , },
{46, 7.63415, pitchIRM(-0.989139), , , , , , , , },
{47, 7.63415, rollIRM(0.989139), , , , , , , , },
{48, 8.63974, pitchRM(0.999124), , , , , , , , },
{49, 8.98187, yF4(-0.396044), yIRM(0.918223), , , , , , , },
{50, 9.89247, yIM(0.707321), yTM(-0.399697), yRM(-0.583015), , , , , , },
{51, 28.1687, rollRM(0.997509), , , , , , , , }

Transfer Functions

Ground displacement (Z) → TM displacement (Z and Pitch)



Ground displacement (Yaw) → TM displacement (Yaw)



IM pushed by actuators on IRM → TM displacement (Z, Longitudinal)